Lecture 3:

Previously on Computing Geometric Structures...

N^3 cpt w/ $\partial N =$ tori

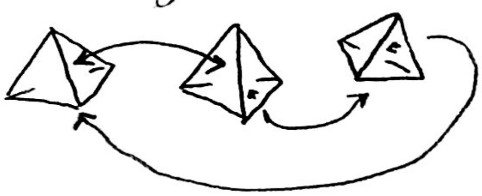
$M = \text{int}(N) = N \setminus \partial N$

Ex:

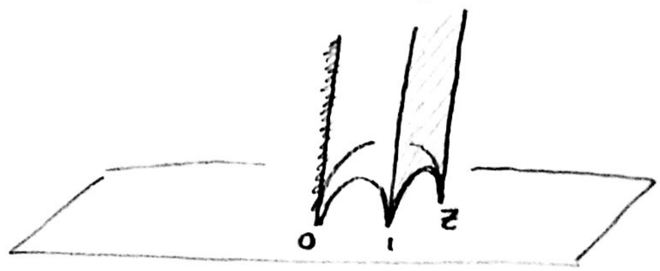
$M = S^3 \setminus \text{Link}$

$M = M_f \quad f \in \text{Mod}(\Sigma_{g,n})$

Ideal triangulation $\mathcal{T}$ of M: cell complex with

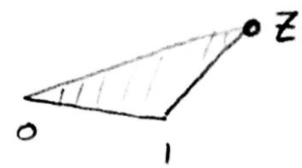


$\mathcal{T} \setminus \mathcal{T}^0 \cong M.$

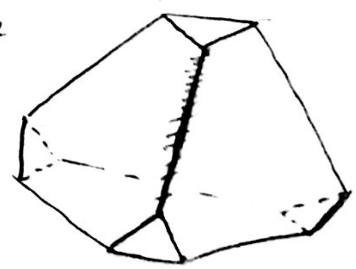


Geodesic ideal tet in $\mathbb{H}^3$:

shape param $z \in \hat{\mathbb{C}} \setminus \{0, 1, \infty\}$ assoc to each edge.



Note: $\mathcal{T} \setminus \mathring{N}(\mathcal{T}^0)$ built from

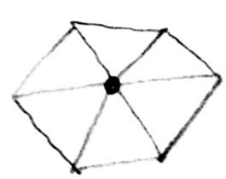


and homeo to N.

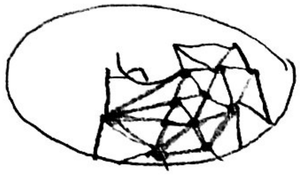
Goal: Find complete hyp str on M by choosing correct shapes z_i for tets in $\mathcal{T}$.

So far: Edge eqns

$z_1 \cdots z_k = 1.$

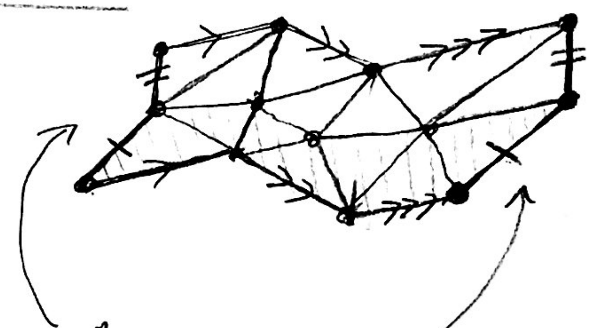


Completeness eqns: Need an $\mathbb{E}^2$ structure on each cusp, only have a similarity str.

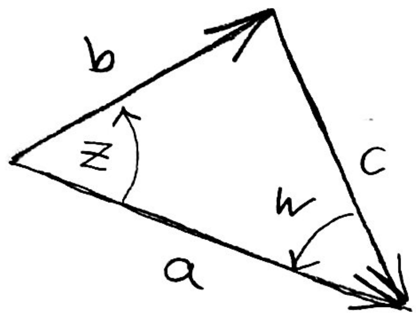


Component of ∂N

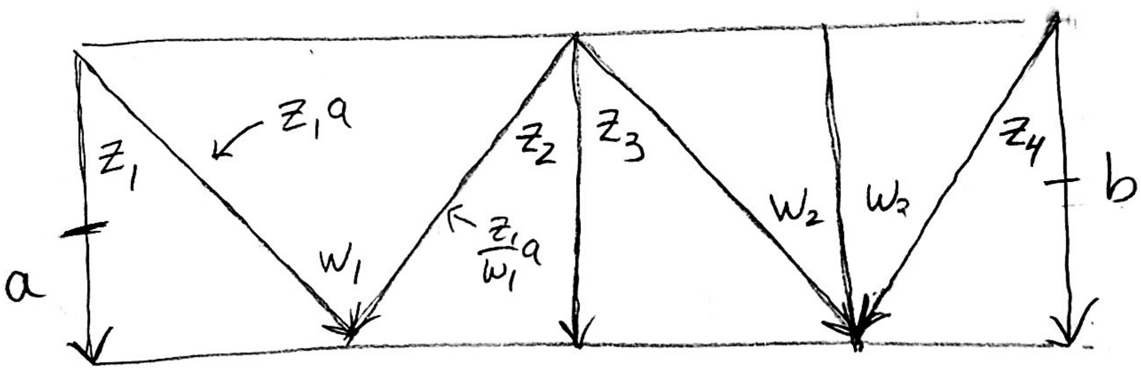
Fund. Domain:



Are these the same vectors?



$$b = z a \qquad -a = w(-c) \iff c = \frac{1}{w} a$$



$$b = \frac{z_1 z_2 z_3 z_4}{w_1 w_2 w_3} a$$

Cusp Eqn: $z_1 z_2 \cdots z_k = w_1 \cdots w_\ell$

Turns out only need two of these.

[Thurston] Suppose $\mathcal{T}$ is an ideal triangulation of M and $z_i \in \mathbb{C} \setminus \{0, 1\}$ satisfy the edge and cusp eqns above <u>and</u>

i) $\operatorname{Im}(z_i) > 0$

ii) For each edge $\sum \arg(z_i) = 2\pi$.

Then these shapes give a complete hyperbolic structure on M with

$$\operatorname{Vol}(M) = \sum \operatorname{Vol}\left(\substack{\text{tet w/} \\ \text{shape } z_i}\right) = \sum \operatorname{Li}_2(z_i)$$

$$< 1.02 (\# \text{ of tet in } \mathcal{T})$$

Bloch-Wigner dilogarithm.

<u>Fact</u>: Ideal tet w/ maximal volume is the one with $z = e^{2\pi i/6}$ , i.e. the regular one.

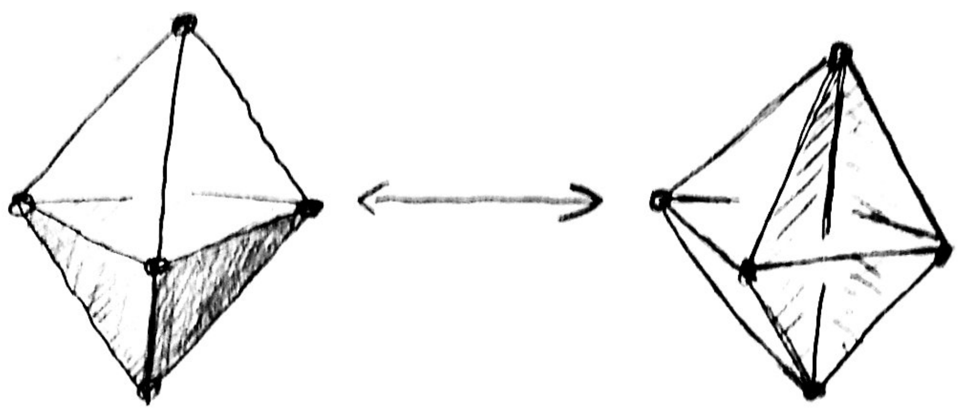
<u>Fact</u>: Because of Mostow, there is at most one such solution.

[Demo: Solutions for random knots.]

Back to homeomorphism problem:

Suppose M_1, M_2 are given as ideal triangulations $\mathcal{T}_1, \mathcal{T}_2$. Are they homeomorphic?

Issue: Both M_i have infinitely many ideal triangulations, all related by



2 tet sharing a face

3 tet around an edge

If $M_1 \cong M_2$ we can eventually prove this by exhaustively enumerating triangulations using these moves.

Assume M_i are hyperbolic as certified by solutions to above eqns for $\mathcal{T}_i$.

Suppose M is a complete hyp n-mfld of finite volume (<u>not</u> cpt).

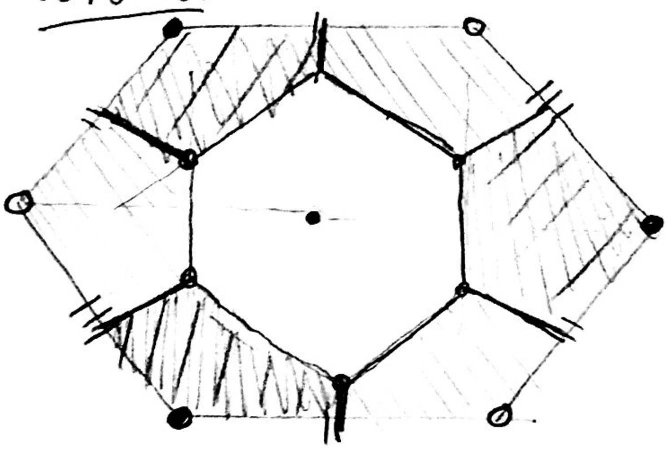
[Epstein-Penner] There is a <u>cannonical ideal cellulation</u> $\mathcal{J}$ of M defined purely from the hyp str on M. The combinatorial symmetries of $\mathcal{J}$ are precisely $\mathrm{Isom}(M)$ when $n \geq 3$.

<u>Strategy for solving homeo prob</u>: Find cannonical cellulations of each M_i. The M_i are homeo $\iff$ these cellulations are combinatorially isomorphic.

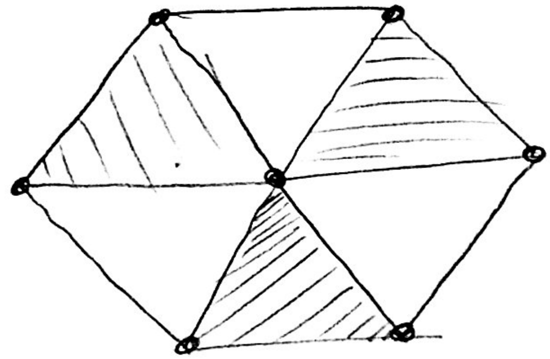
<u>Geometry to cellulations</u>:

Consider a Euclidean 2-torus. Pick some pts and consider the resulting Voronoi partition.

Voronoi



Delaunay: (dualize!)



Works great for any geometry and dimension
Not very cannonical though.

Idea: Use the cusps to define the Voronoi partion, called the Ford domain in this context. Problem: cusps are infinitely far away...

Cartoon: One cusped surface