Lecture 2:

Last time: "Topology = Geometry"
in dimension 3

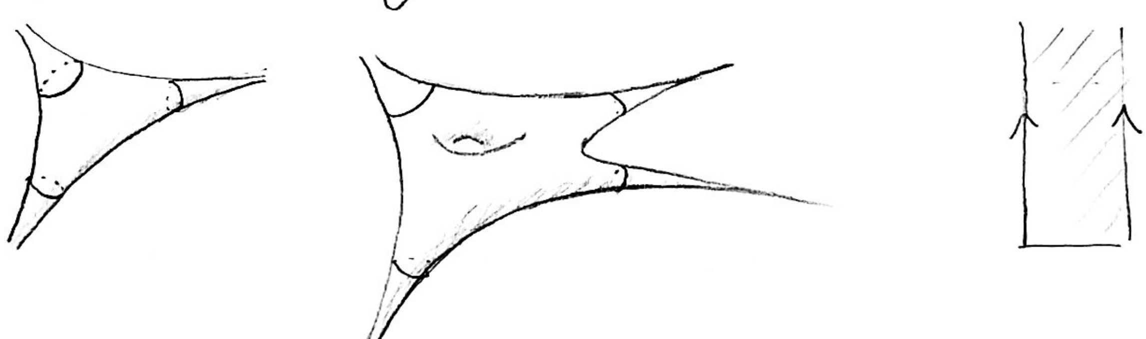
Focus: Complete hyperbolic 3-mflds of finite volume

Ex: $S^3 \setminus$  $= \Gamma \backslash \mathbb{H}^3$

$$\Gamma = \left\langle \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ \zeta & 1 \end{pmatrix} \right\rangle \quad \zeta = e^{2\pi i/6}$$

Today: Finding hyperbolic structures

Warmup: Complete hyperbolic surfaces of finite area.



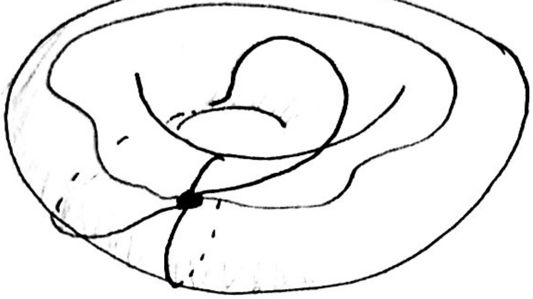
$z \longmapsto z+1$

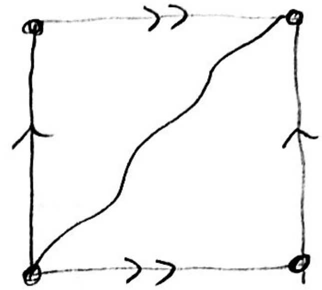
Ideal triangulation:

Built from triangles w/ sides identified so that vertices = punctures. Just a topological notion.

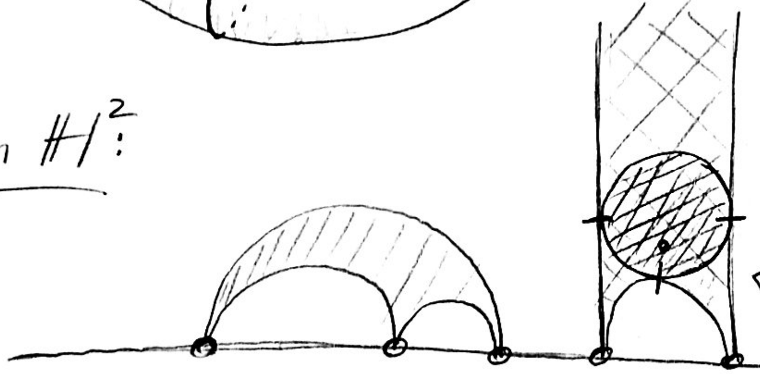
$$g_{\mathbb{H}^2} = \frac{1}{y^2} g_{E^2}$$

Ex:



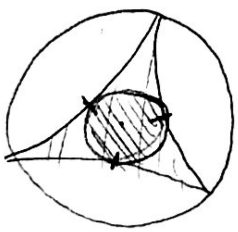


In $\mathbb{H}^2$:

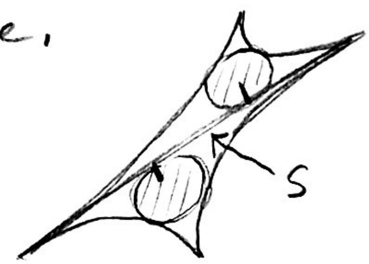


Unique geodesic ideal triangle up to isometry.

isometric to $\mathbb{R}$.

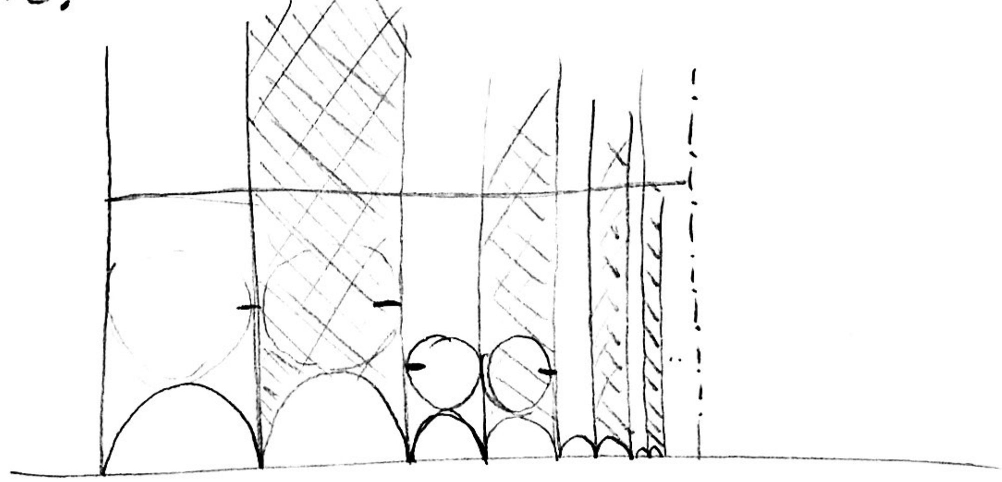


To put a hyperbolic structure on a ideal triangulation need to choose a shear for each edge.



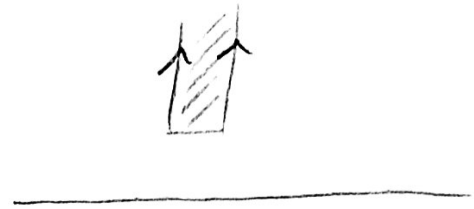
Any choice gives a hyperbolic metric on the surface. Warning: it may not be complete!

Ex:

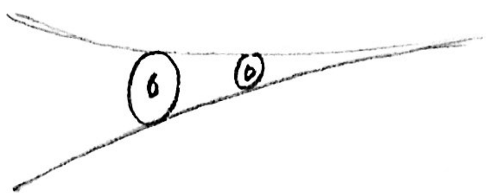


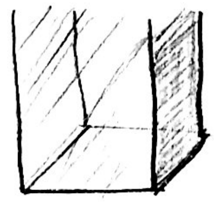
[Problem sheet explores this in detail...]

Cusps: 2-d



3-d

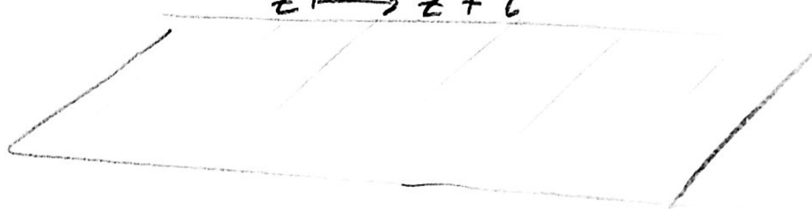




$\circ\ (x, y, t)$

$z \mapsto z + 1$
$z \mapsto z + i$

Check: Volume is finite.



Fact: Any complete hyp. 3-mfld M of finite volume has ends of this type.

$$g_{\mathbb{H}^3} = \frac{1}{t^2} g_{\mathbb{E}^3}$$

$$\mathrm{Isom}^+(\mathbb{H}^3) \cong \mathrm{M\ddot{o}b}(\hat{\mathbb{C}}) \cong PSL_2\mathbb{C}$$

In particular $M = \overbrace{N \setminus \partial N}^{\mathrm{int}(N)}$

where N is cpt with ∂N a union of tori.

Suppose $M = \mathrm{int}(N)$ as above. An ideal triangulation $\mathcal{J}$ of M is a cell complex built from finitely many tetrahedra by gluing faces in pairs such that

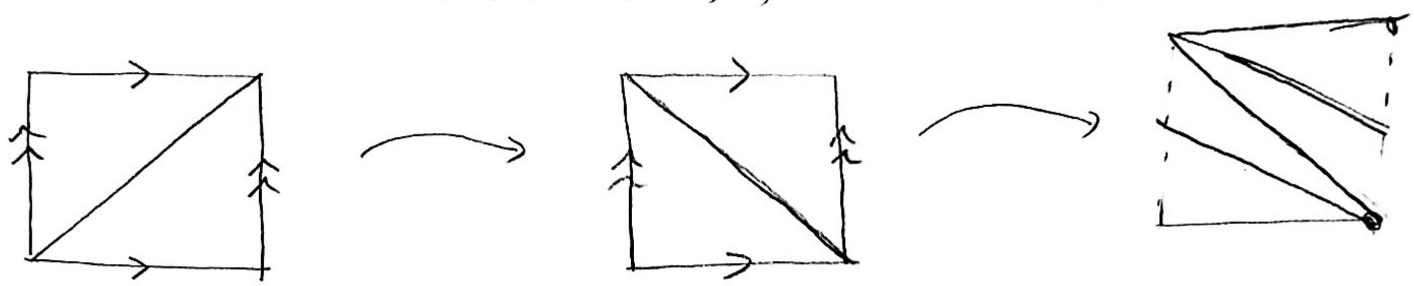
$\mathcal{J} \setminus \mathcal{J}^0 \cong M$

Ex: $S^3 \setminus$  has an ideal triangulation with 2 tetrahedra.

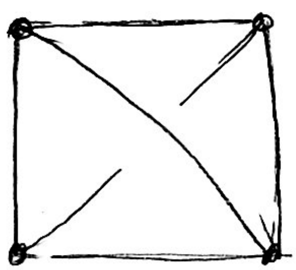
Ex: $f \in \mathrm{Mod}(\Sigma_{g,n})$

$$M_f = \Sigma \times I \Big/ (x,1) \sim (f(x),0)$$

[M_f hyperbolic $\Leftrightarrow$ f is pA.]



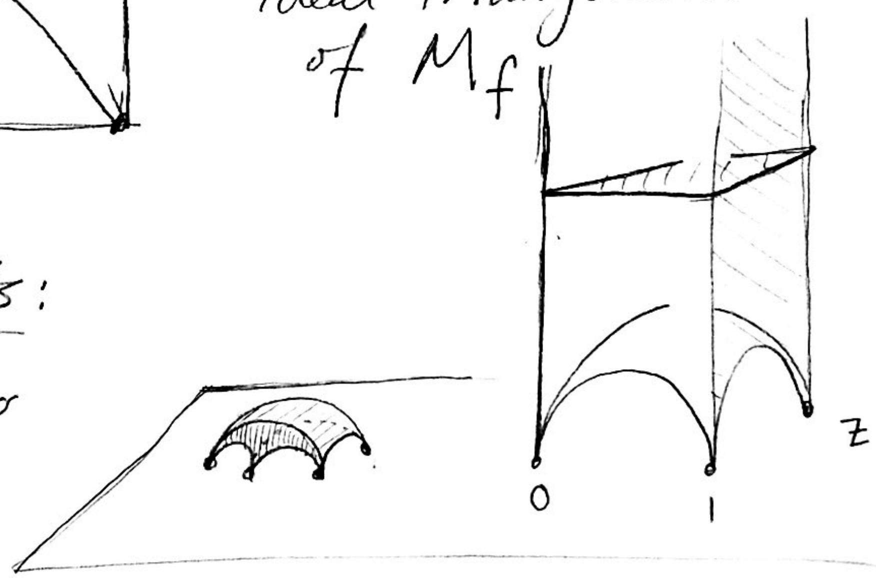
Can imple. a flip by gluing in a tetrahedra.



Can use to give an ideal triangulation of M_f

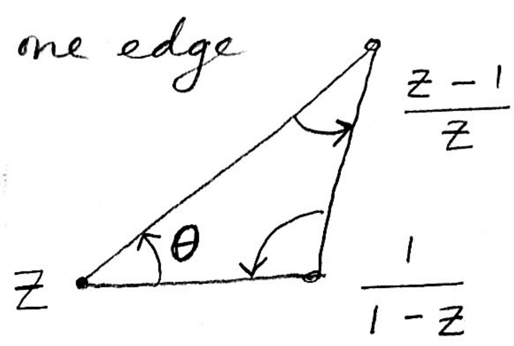
Geometric ideal tets:

moduli = cross ratio



Call z the shape param. associated to the edge joining 0 to ∞. [think "complex dihedral angle"]

Note: The shape param of any one edge determines the others.



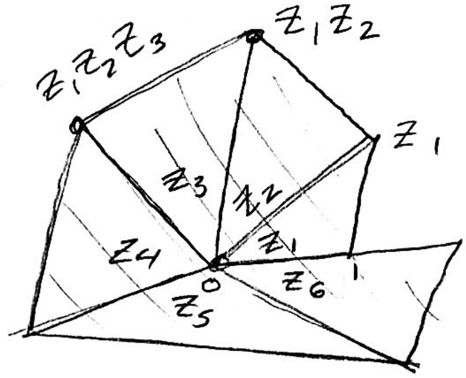
$$z = re^{i\theta}$$

Setting: $\mathcal{J}$ ideal triangulation

z_i = shape of tet $i \in \mathbb{C} \setminus \{0, 1, \infty\}$

Edge equations: [Motivation: Σ dihedral angles $= 2\pi$]

Looking down an edge



$$z_1 z_2 \cdots z_k = 1.$$

Deformation Variety:

$$D(\mathcal{J}) = \left\{ \vec{z} \in \left(\mathbb{C} \setminus \{0,1,\infty\}\right)^n \;\middle|\; \text{all edge eqns sat.} \right\}$$

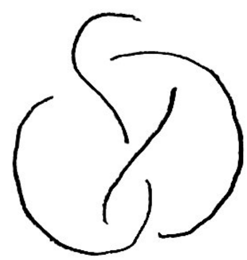
Fact: After removing redundancies, there are $n-1$ equations. So expect $\dim_{\mathbb{C}} D(\mathcal{J}) = 1$.

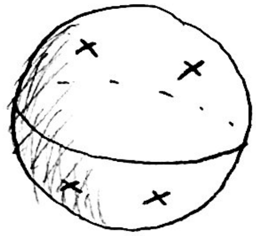
Any point in $D(\mathcal{J})$ gives a hyperbolic structure on M. [May not be complete!]

Have a map

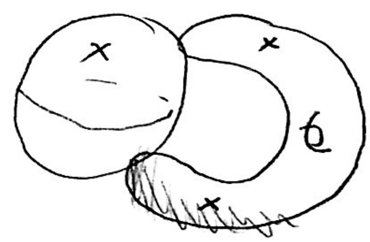
$$\mathcal{D}(J) \longrightarrow \bar{X}(M) = \mathrm{Hom}(\pi_1 M, PSL_2\mathbb{C}) /\!\!/_{PSL_2\mathbb{C}}$$

Combinatorial developing map $\rightsquigarrow$ holonomy rep'n.

<u>Ex</u>: $M = S^3 \setminus$  has a 2-tet triangulation

$\mathcal{D}(J) =$ 

$X(M)$



$\bar{X}(M) =$