Warwick Research School: Lecture 1.

Geometric Topology: study of manifolds.

Convention: All manifolds will be orientable.

Homeo Prob: Given two closed n-mflds M and N (say as simplical complexes) are they homeomorphic? Is this decidable? [i.e. $\exists$? a computer algorithm]

$\underline{n=1}$: Yes   return "yes"

$\underline{n=2}$: Yes   return $\chi(M) = \chi(N)$.

$\underline{n \geq 4}$: No   [Markov 1958] "manifolds are at least as complicated as finitely presented groups."

$\underline{n=3}$: Yes   Focus of these lectures.

Role of Geometry:

$\underline{n=2}$: Any closed surface has a const. curve metric.

$\underline{n \geq 4}$: Homogenous geometry is "rare".

$\underline{n=3}$: Some have no const. curve metrics: $S^2 \times S^1$

Reason: Univ cover not homeo to S^3, $\mathbb{E}^3$, $\mathbb{H}^3$

Geometrization Thm [Thurston, Perelman, ...] Any closed 3-manifold has a decomposition along essential  and  into pieces w/ metrics modelled on one of: S^3, $\mathbb{E}^3$, $\mathbb{H}^3$, $S^2 \times \mathbb{R}$, $\mathbb{H}^2 \times \mathbb{R}$, Nil, $Solv$, $\widetilde{SL_2 \mathbb{R}}$

A M^3 is *prime* if $M = N_1 \# N_2 \Rightarrow$ some $N_i \cong S^3$.

Note: Any topological M^3 has a unique smooth structure.
[So can read $\cong$ as homeo or diffeo.]

[Kneser-Milnor] Any closed $M^3 = N_1 \# \cdots \# N_k$ with N_i prime. Unique up to perm. the N_i.

A closed surface $S \neq \bigcirc$ in M^3 is *incompressible/essential* if $\pi_1 S \hookrightarrow \pi_1 M$.

[JSJ] Torus decomposition.

$\{|x|<1\} =$

Most important/common geometry: $\mathbb{H}^3 =$

$$g_{\mathbb{H}^3} = \frac{4}{(1-r^2)^2} g_{\mathbb{E}^3}$$

[Mflds w/ other geoms are classified...]

A *hyperbolic structure* on M^3 is a Riem. metric of const curve -1. Equivalently,

$$M = {}_{\Gamma}\backslash^{\mathbb{H}^3} \qquad \Gamma = \pi_1 M \leq \mathrm{Isom}^+(\mathbb{H}^3)$$

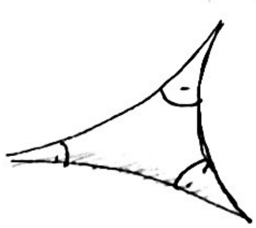
discrete, torsion free

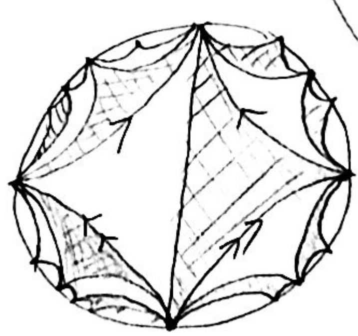
$$\mathrm{Isom}^+(\mathbb{H}^3) = \text{Möb}^+(\hat{\mathbb{C}}) = PSL_2\mathbb{C}.$$

Motivating example:

$$\Lambda = \left\langle \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} \right\rangle$$

${}_{\Lambda}\backslash^{\mathbb{H}^2} =$





Not compact but area $= 2\pi$

$$\Gamma = \left\langle \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 2-i & 1 \\ 2i & i \end{pmatrix}, \begin{pmatrix} 3 & 2i \\ 2i & -1 \end{pmatrix} \right\rangle$$

$$M = {}_{\Gamma}\backslash^{\mathbb{H}^3} = S^3 \setminus$$  Borromean Rings $= 6^3_2$

Volume $\approx 7.327724753\ldots$

<u>Mostow Rigidity</u>: Suppose M and N are hyperbolic n-manifolds of finite volume where $n \geq 3$. If $\pi_1 M \cong \pi_1 N$ then M and N are <u>isometric</u>.

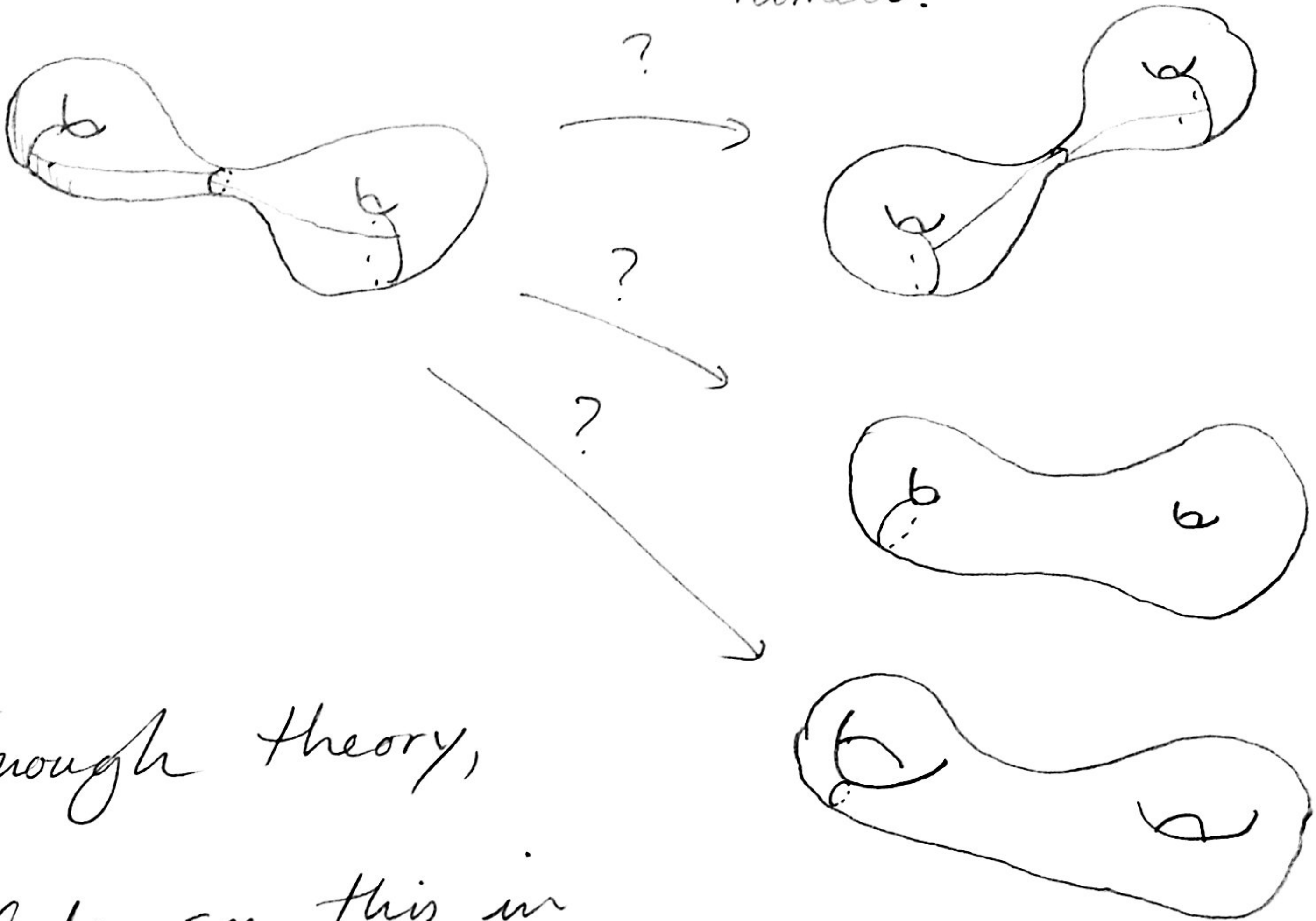
<u>Cor</u>: Any geometric invariant of a hyp. n-mfld for $n \geq 3$ is a topological invariant.

Thurston's Mantra: "Topology = Geometry" in dim 3.

How this connects to the homeo. problem, see [Kuperberg 2017].

1. Find the geom. decomp.
2. Non-hyp pieces are classified
3. For hyp. pieces need check for an <u>isometry</u>.

Much easier to check for isometries than homeos.



Enough theory, let's see this in practice!

SnapPy demo.