A unified Casson-Lin invariant for the real forms of SL(2)

Nathan Dunfield (University of Illinois)

Joint with Jake Rasmussen

Based on arXiv:2209.03382

Notes already posted at:
https://dunfield.info/tech2023.pdf

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Much learned about 3-manifolds by studying reps $\pi_1 M^3 \to G$ for G one of:

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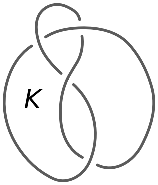

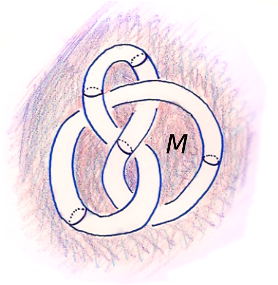

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For $\theta \notin D_M$, can define

$$h^\theta_{\mathrm{SU}_2}(M) = \text{signed count of } X^{\theta,\mathrm{irr}}_{\mathrm{SU}_2}(M)$$

Moreover, $h^\theta_{\mathrm{SU}_2}(M) = -\frac{1}{2}\sigma_K(e^{i2\theta})$, which is constant outside of D_M.

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[Kronheimer-Mrowka] A nontrivial K has an irred $\rho\colon \pi_1 M \to \mathrm{SU}_2$.

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Let $\Sigma_n(K)$ be the n-fold cyclic cover of S^3 branched over K.

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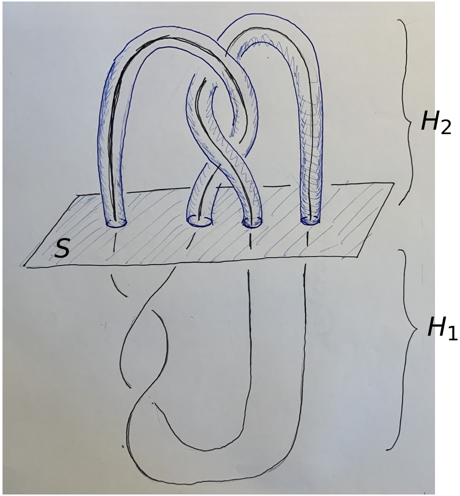
Casson-Lin signs. Write $M = H_1 \cup_S H_2$ using an n-bridge diagram for K:



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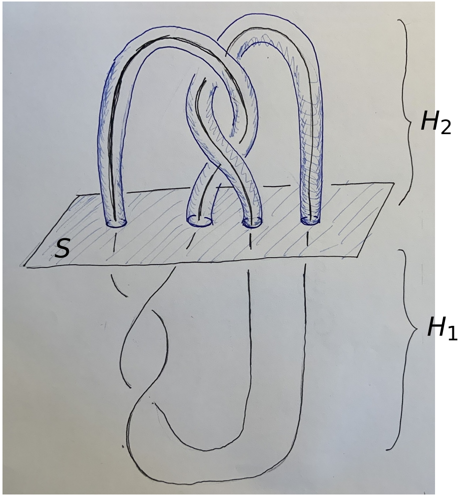

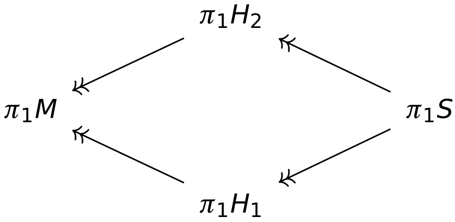

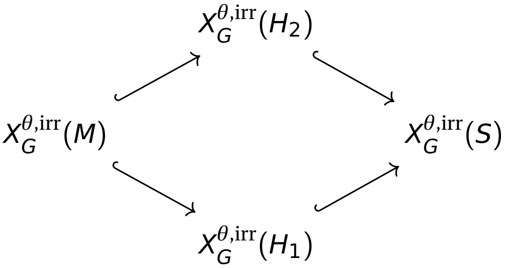

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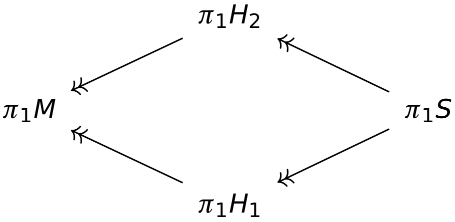

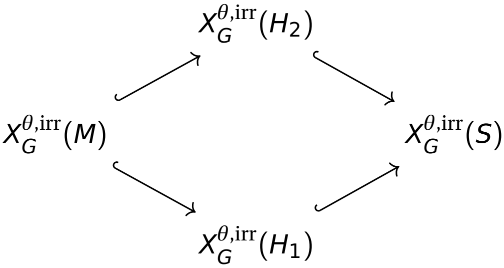

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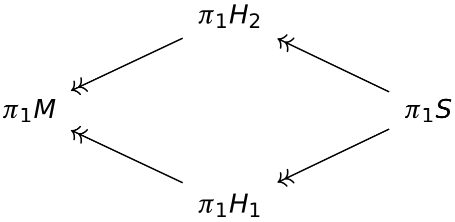

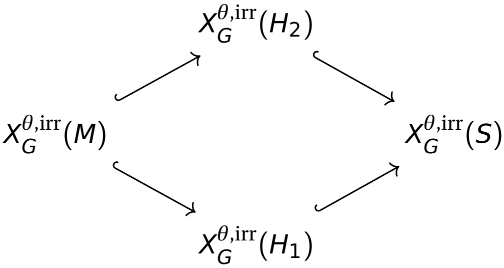

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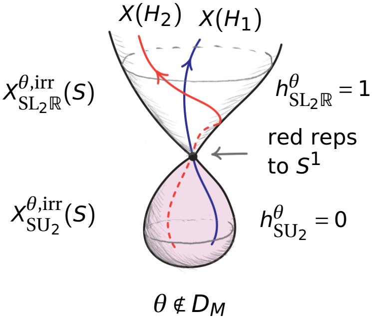

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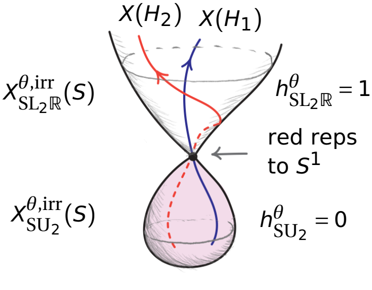

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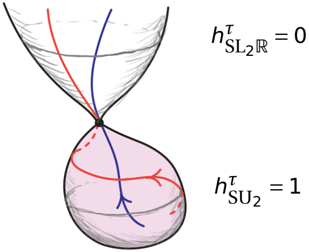

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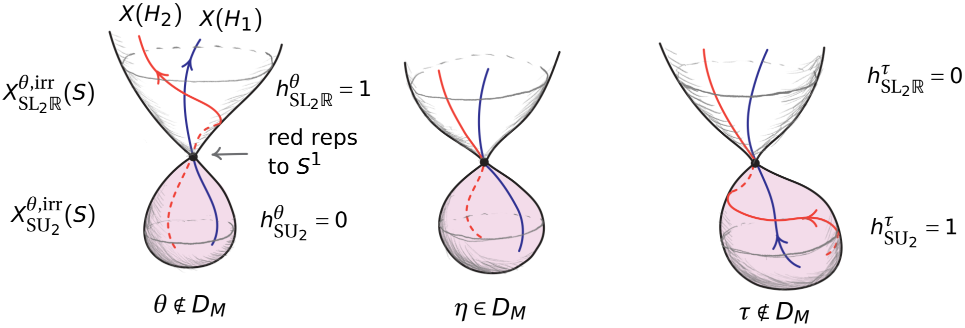
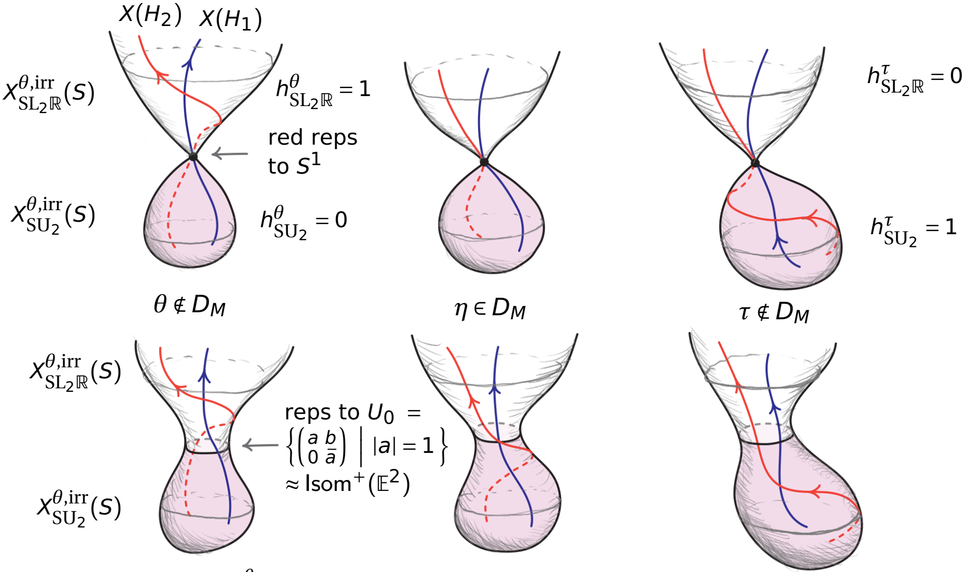


Unification: look at inside $X^{\theta,\mathrm{irr}}_{\mathrm{SL}_2\mathbb{C}}(S)$.



Resolution $\mathscr{X}^{\theta}(S)$

Moral: in resolved picture $h(M)$ is the alg $\cap\#$ of red and blue for **all** angles.