

The Least Spanning Area of a Knot and the Optimal Bounding Chain Problem

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Knot in \mathbb{R}^3 : Smooth embedding of S^1 in \mathbb{R}^3 .



Spanning surface: Any knot in \mathbb{R}^3 is the boundary of a smooth orientable embedded surface S .

Knot Genus: What is the least genus of such an S ?



Least Spanning Area: What is the least area of such an S ?

Both questions are decidable [Haken 1960, Sullivan 1990].

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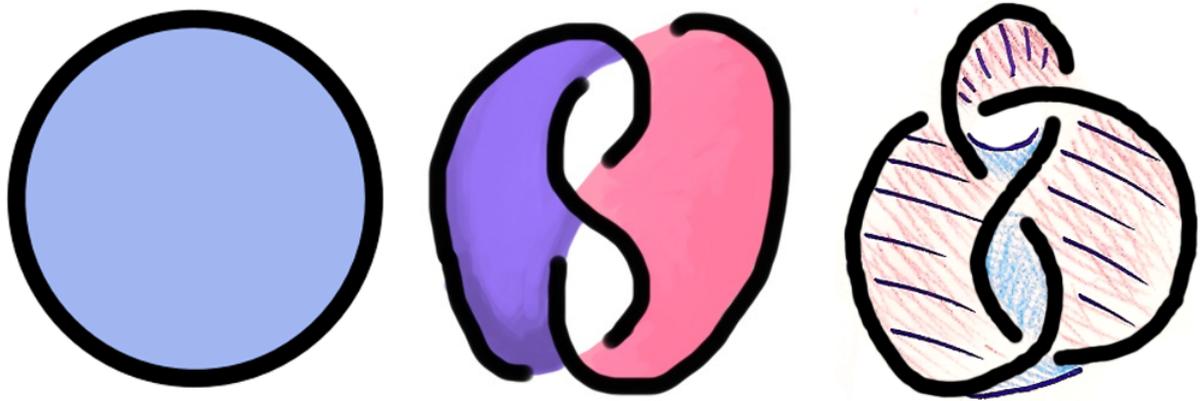
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More generally, consider a closed orientable 3-manifold Y containing a knot K .

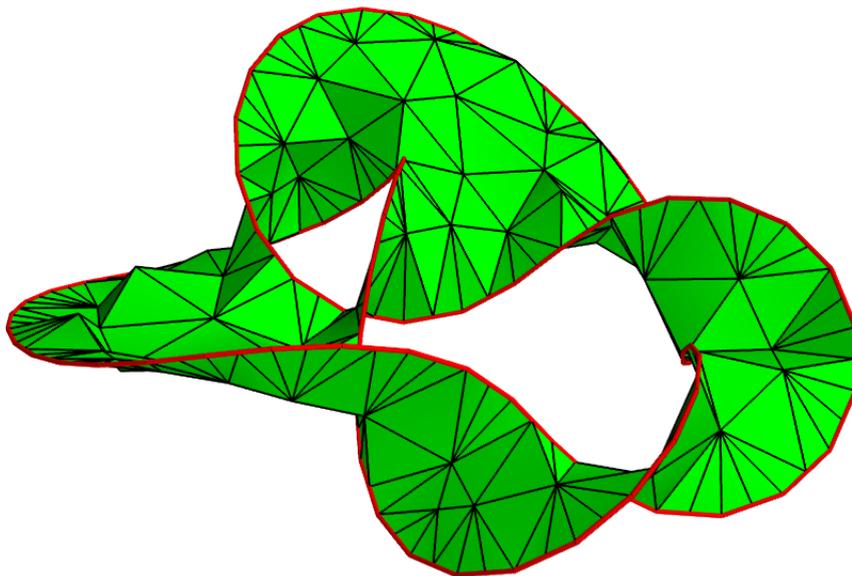
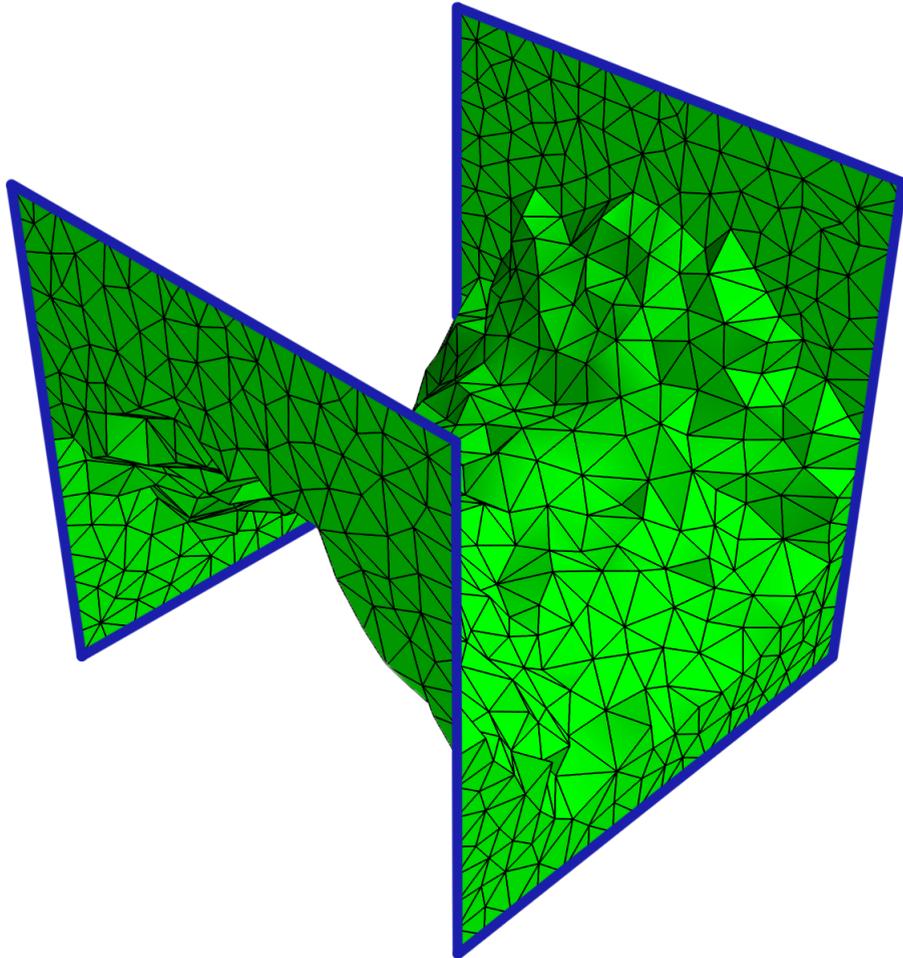
- Y is given as a simplicial complex \mathcal{T} with areas (in \mathbb{N}) assigned to each 2-simplex.
- K is a loop of edges in \mathcal{T} .
- Consider spanning surfaces which are “made out of” 2-simplices of \mathcal{T} .

Agol-Hass-Thurston (2002) *For general Y the Knot Genus and Least Spanning Area problems are NP-hard.*

Thm (D-H) *When $H_2(Y; \mathbb{Z}) = 0$, e.g. $Y = S^3$, Least Spanning Area can be solved in polynomial time.*

Conj *When $H_2(Y; \mathbb{Z}) = 0$, Knot Genus can be solved in polynomial time.*

Algorithm uses linear programming.

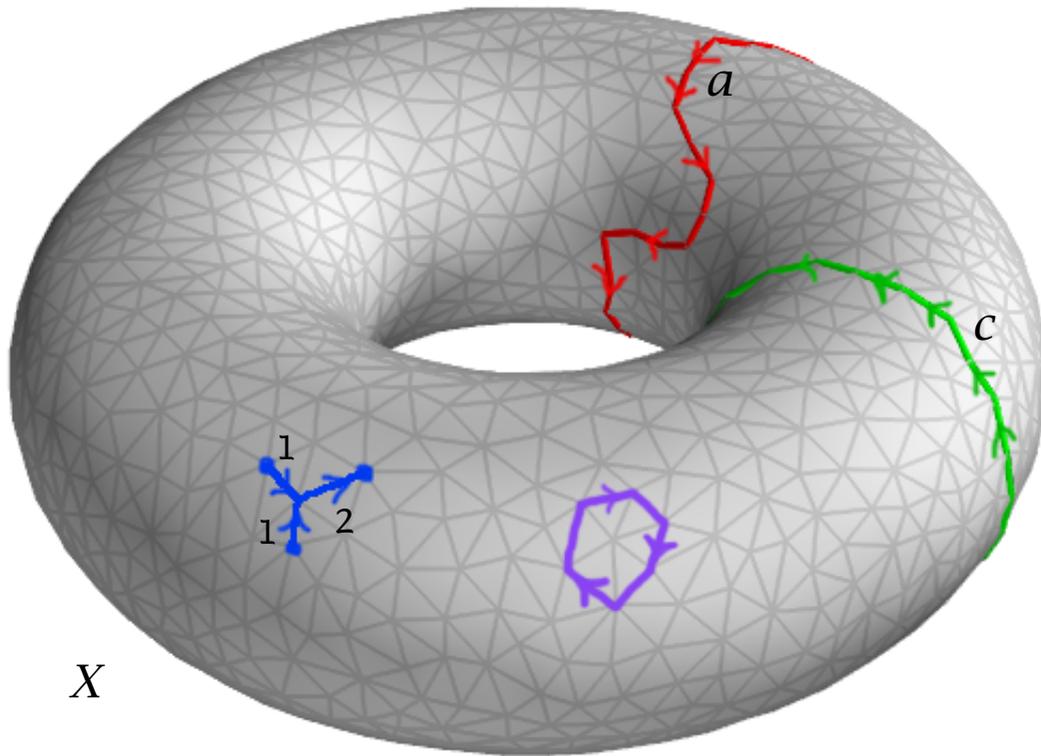


Thm (D-H) *When $H_2(Y; \mathbb{Z}) = 0$, e.g. $Y = S^3$, Least Spanning Area can be solved in polynomial time.*

Approach:

1. Consider the related Optimal Bounding Chain Problem, where S is a union of 2-simplices of \mathcal{T} but perhaps not a surface.
2. Reduce to an instance of the Optimal Homologous Chain Problem that can be solved in polynomial time by [Dey-H-Krishnamoorthy 2010].
3. Desingularize the result using two topological tools.

Homology: X a finite simplicial complex, with $C_n(X; \mathbb{Z})$ the free abelian group with basis the n -simplices of X .



Boundary map: $\partial_n: C_n(X; \mathbb{Z}) \rightarrow C_{n-1}(X; \mathbb{Z})$

Homology:

$$\begin{aligned}
 H_n(X; \mathbb{Z}) &= \ker(\partial_n) / \text{image}(\partial_{n+1}) \\
 &= \frac{\{n\text{-dim things without boundary}\}}{\{\text{boundaries of } (n+1)\text{-dim things}\}}
 \end{aligned}$$

Example: $H_1(\text{torus}) = \mathbb{Z}^2$.

A knot K in an orientable 3-manifold Y gives an element of $H_1(Y; \mathbb{Z})$; when this is zero, K has a spanning surface by Poincaré-Lefschetz duality. Thus if $H_1(Y; \mathbb{Z}) = 0$, e.g. $Y = S^3$ or \mathbb{R}^3 , then every knot has a spanning surface.



Assign a “volume” to each n -simplex in X , which gives $C_n(X; \mathbb{Z})$ an ℓ^1 -norm.

Optimal Homologous Chain Problem (OHCP)

Given $a \in C_n(X; \mathbb{Z})$ find $c = a + \partial_{n+1}x$ with $\|c\|_1$ minimal.

Optimal Bounding Chain Problem (OBCP)

Given $b \in C_{n-1}(X; \mathbb{Z})$ which is 0 in $H_{n-1}(X; \mathbb{Z})$, find $c \in C_n(X; \mathbb{Z})$ with $b = \partial_n c$ and $\|c\|_1$ minimal.

Thm (D-H) *Both OHCP and OBCP are NP-hard.*

OHCP with mod 2 coefficients is **NP**-complete by [Chen-Freedman 2010].

Dey-H-Krishnamoorthy (2010) *When X is relatively torsion-free in dimension n , then the OHCP for $C_n(X; \mathbb{Z})$ can be solved in polynomial time.*

Key: Orientable $(n + 1)$ -manifolds are relatively torsion-free.

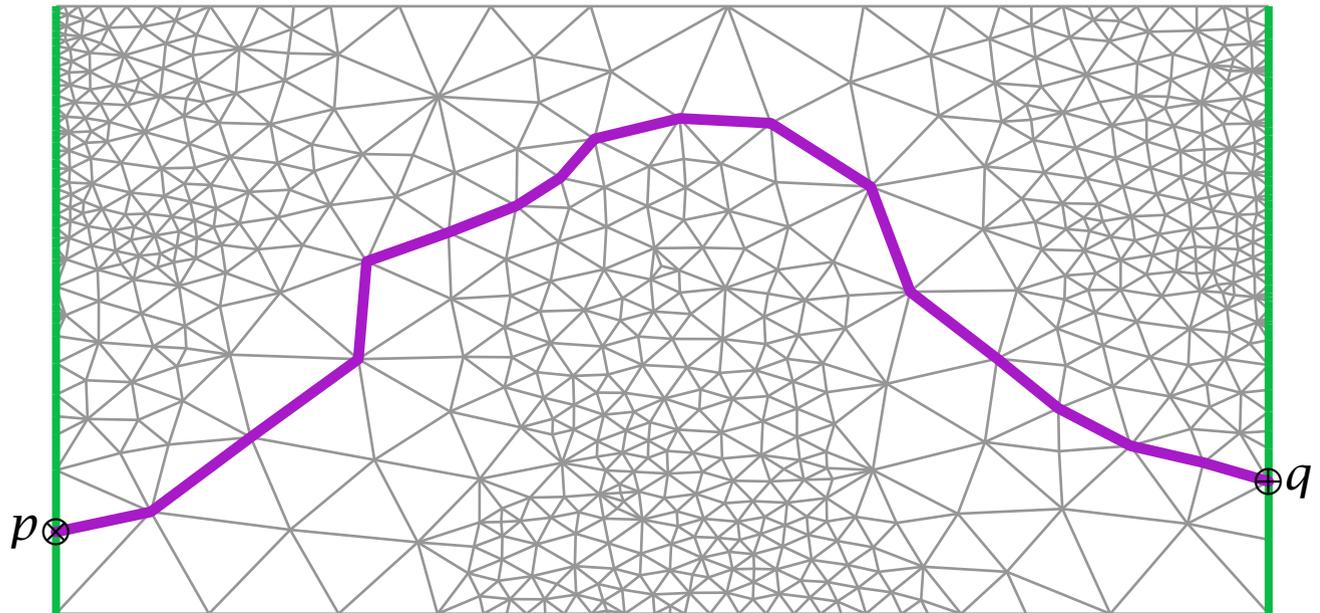
Thm (D-H) *When X is relatively torsion free in dimension n and $H_n(X; \mathbb{Z}) = 0$, then the OBCP for $C_{n-1}(X; \mathbb{Z})$ can be solved in polynomial time.*

Compare

Thm (D-H) *When $H_2(Y; \mathbb{Z}) = 0$, the Least Spanning Area problem for a knot K can be solved in polynomial time.*

Desingularization: a toy problem

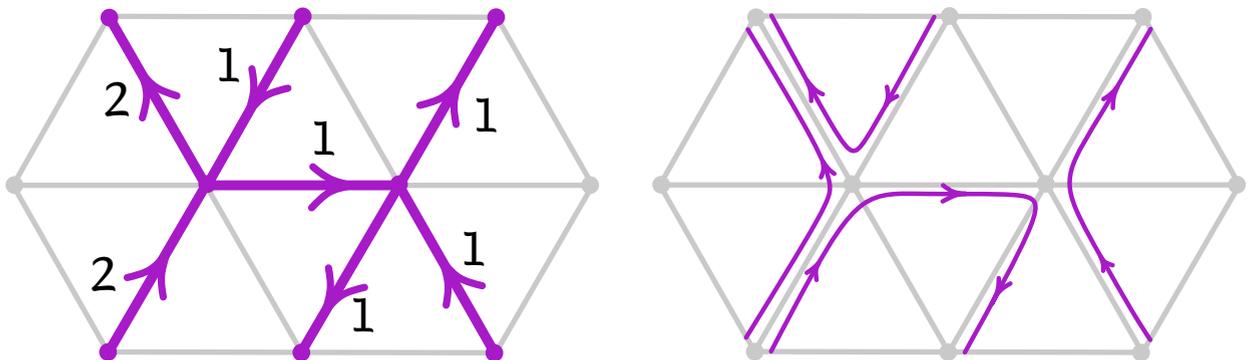
In a triangulated rectangle X , find the shortest embedded path in the 1-skeleton joining vertices p and q .



Consider $b = q - p \in C_0(X; \mathbb{Z})$, which is 0 in $H_0(X; \mathbb{Z})$.

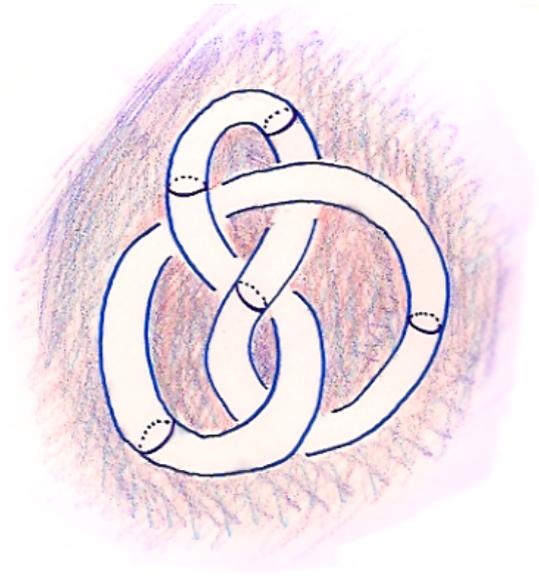
Let $c \in C_1(X; \mathbb{Z})$ be a solution to the OBCP for b .

Claim: c corresponds to an embedded simplicial path.



Rest of desingularization

1. Pass to the exterior of the knot K .



2. Introduce a relative version of the Optimal Bounding Chain Problem.