The Least Spanning Area of a Knot and the Optimal Bounding Chain Problem

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Knot in $\mathbb{R}^3$: Smooth embedding of S^1 in $\mathbb{R}^3$.

Spanning surface: Any knot in $\mathbb{R}^3$ is the boundary of a smooth orientable embedded surface S.

Knot Genus: What is the least genus of such an S?

Least Spanning Area: What is the least area of such an S?

Both questions are decidable [Haken 1960, Sullivan 1990].

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More generally, consider a closed orientable 3-manifold Y containing a knot K.

- Y is given as a simplicial complex $\mathcal{T}$ with areas (in $\mathbb{N}$) assigned to each 2-simplex.
- K is a loop of edges in $\mathcal{T}$.
- Consider spanning surfaces which are "made out of" 2-simplices of $\mathcal{T}$.

Agol-Hass-Thurston (2002) *For general Y the Knot Genus and Least Spanning Area problems are **NP**-hard.*

Thm (D-H) *When $H_2(Y;\mathbb{Z}) = 0$, e.g. $Y = S^3$, Least Spanning Area can be solved in polynomial time.*

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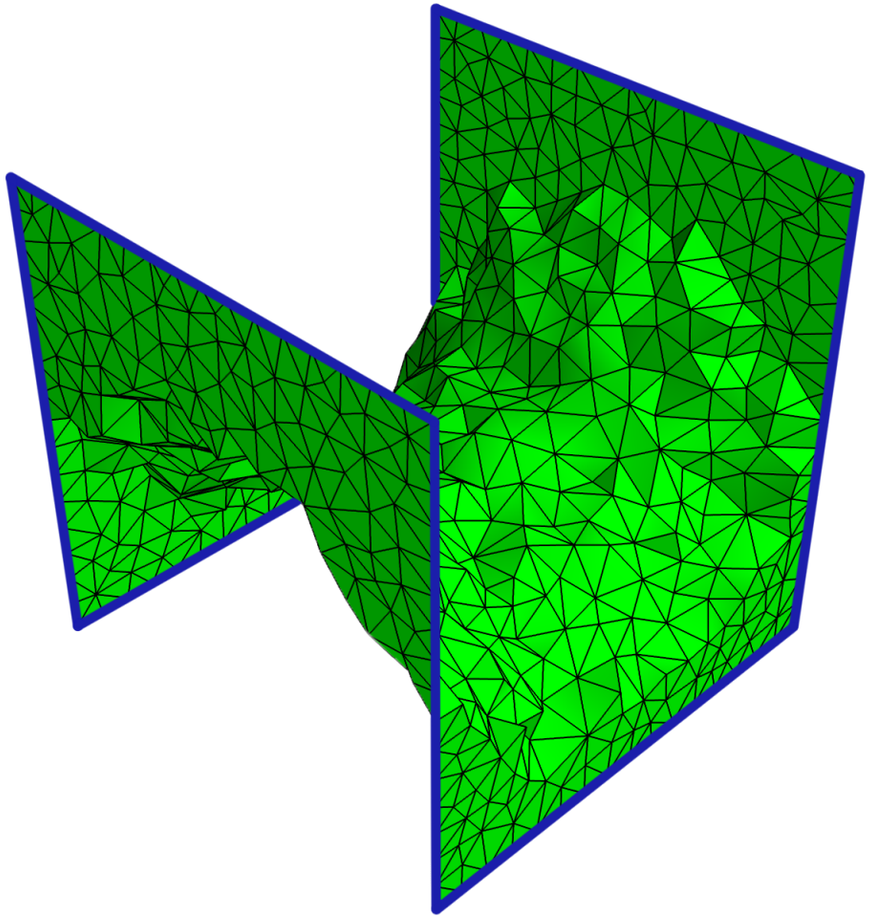

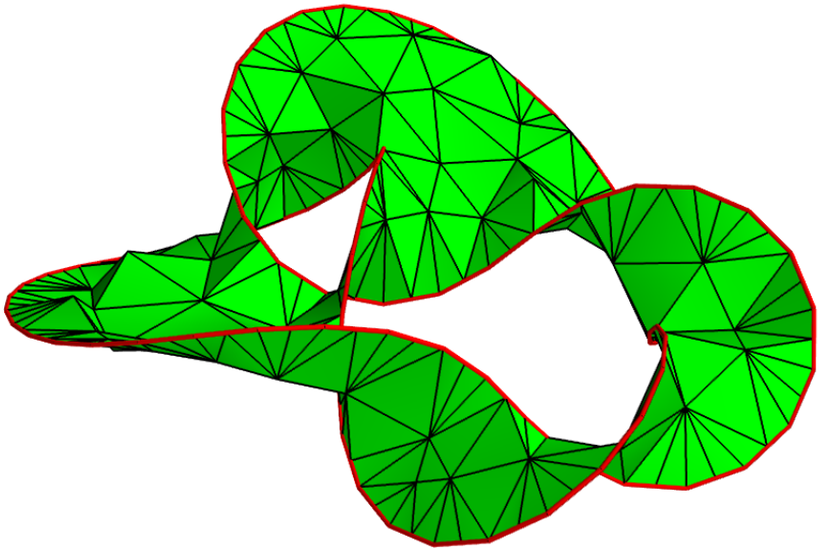

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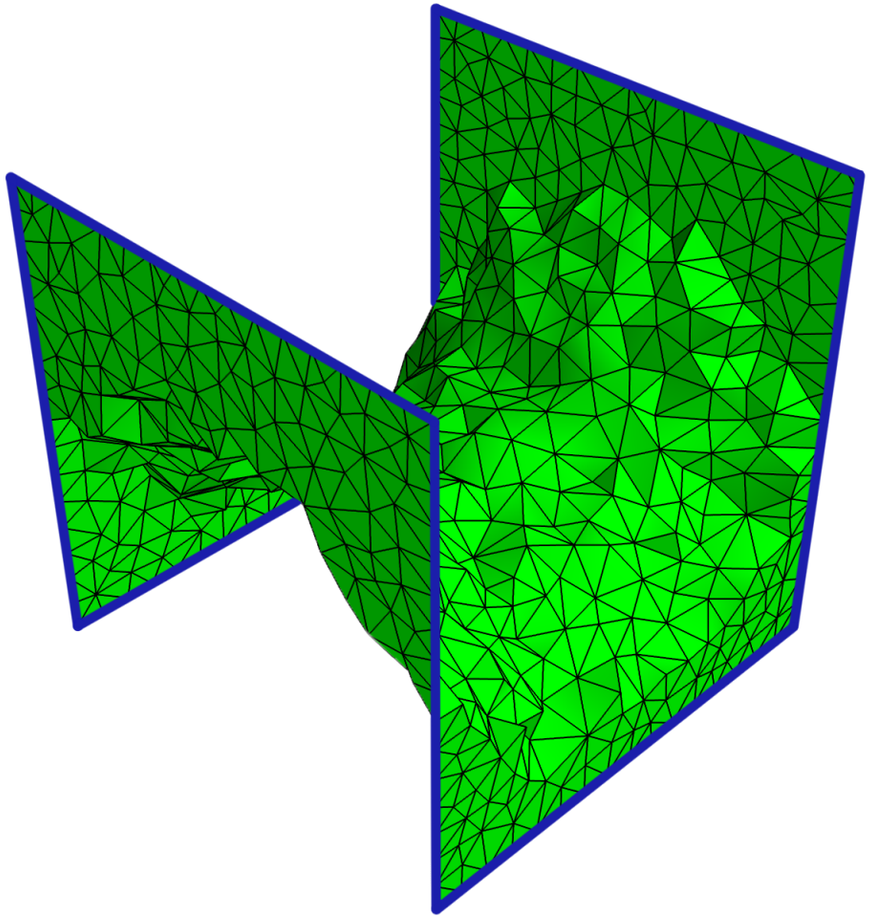

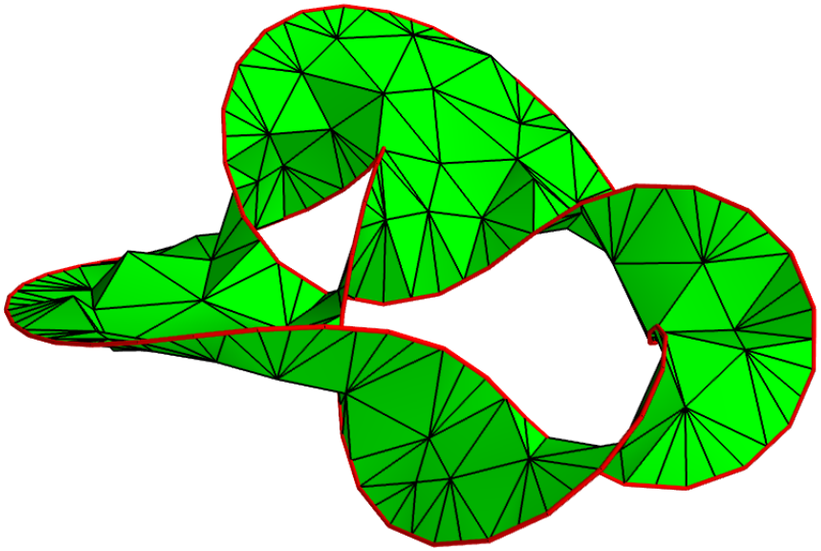

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Approach:

1. Consider the related Optimal Bounding Chain Problem, where S is a union of 2-simplices of $\mathcal{T}$ but perhaps not a surface.
2. Reduce to an instance of the Optimal Homologous Chain Problem that can be solved in polynomial time by [Dey-H-Krishnamoorthy 2010].
3. Desingularize the result using two topological tools.

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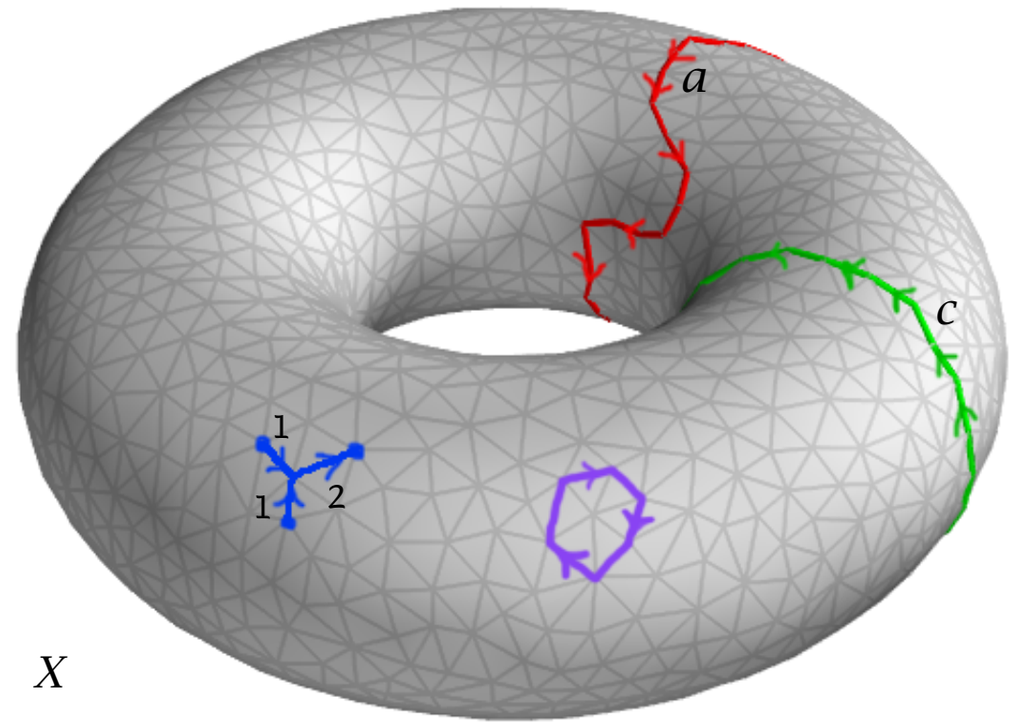
Homology: X a finite simplicial complex, with $C_n(X;\mathbb{Z})$ the free abelian group with basis the n-simplices of X.



Boundary map: $\partial_n\colon C_n(X;\mathbb{Z}) \to C_{n-1}(X;\mathbb{Z})$

Homology:

$$H_n(X;\mathbb{Z}) = \ker(\partial_n) \big/ \mathrm{image}(\partial_{n-1})$$
$$= \frac{\{n\text{-dim things without boundary}\}}{\{\text{boundaries of } (n+1)\text{-dim things}\}}$$

Example: $H_1(\text{torus}) = \mathbb{Z}^2$.

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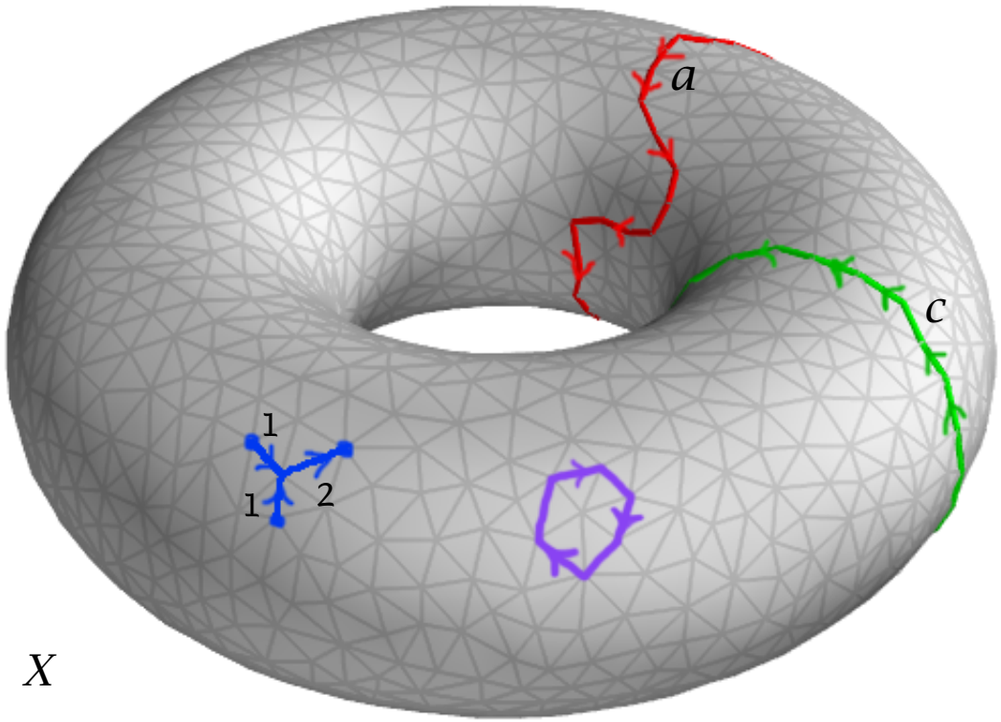

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A knot K in an orientable 3-manifold Y gives an element of $H_1(Y;\mathbb{Z})$; when this is zero, K has a spanning surface by Poincaré-Lefschetz duality. Thus if $H_1(Y;\mathbb{Z}) = 0$, e.g. $Y = S^3$ or $\mathbb{R}^3$, then every knot has a spanning surface.

Assign a "volume" to each n-simplex in X, which gives $C_n(X;\mathbb{Z})$ an ℓ^1-norm.

Optimal Homologous Chain Problem (OHCP)
Given $a \in C_n(X;\mathbb{Z})$ find $c = a + \partial_{n+1}x$ with $\|c\|_1$ minimal.

Optimal Bounding Chain Problem (OBCP)
Given $b \in C_{n-1}(X;\mathbb{Z})$ which is 0 in $H_{n-1}(X;\mathbb{Z})$, find $c \in C_n(X;\mathbb{Z})$ with $b = \partial_n c$ and $\|c\|_1$ minimal.

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Thm (D-H) *Both OHCP and OBCP are **NP**-hard.*

OHCP with mod 2 coefficients is **NP**-complete by [Chen-Freedman 2010].

Dey-H-Krishnamoorthy (2010) *When X is relatively torsion free in dimension n, then the OHCP for $C_n(X;\mathbb{Z})$ can be solved in polynomial time.*

Key: Applies when X is an orientable $n+1$ manifold.

Thm (D-H) *When X is relatively torsion free in dimension n and $H_n(X;\mathbb{Z}) = 0$, then the OBCP for $C_{n-1}(X;\mathbb{Z})$ can be solved in polynomial time.*

Compare

Thm (D-H) *When $H_2(Y;\mathbb{Z}) = 0$, the Least Spanning Area problem for a knot K can be solved in polynomial time.*

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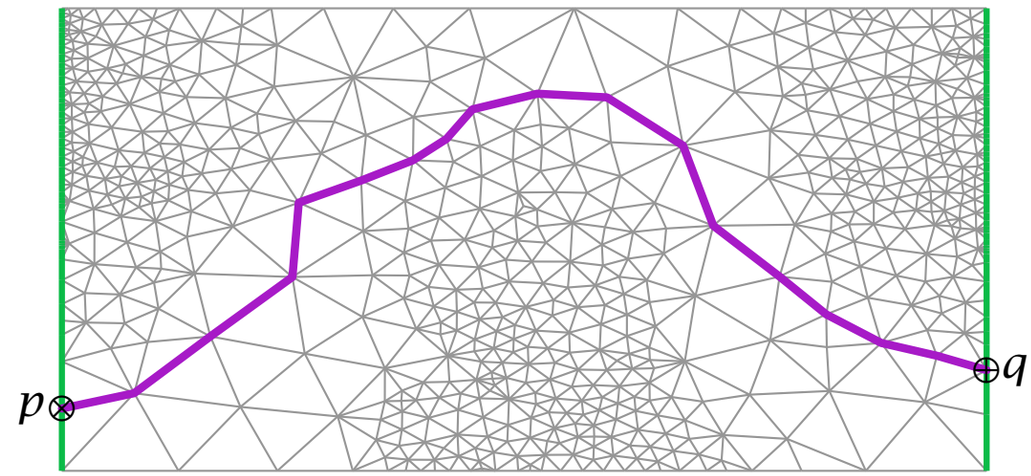
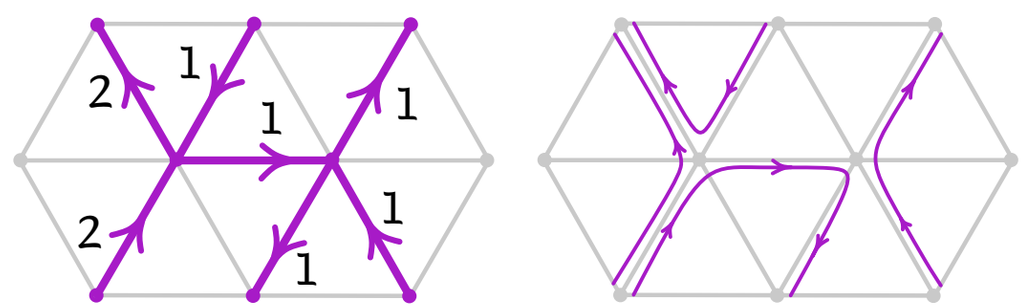
Desingularization: a toy problem

In a triangulated rectangle X, find the shortest embedded path in the 1-skeleton joining vertices p and q.



Consider $b = q - p \in C_0(X;\mathbb{Z})$, which is 0 in $H_0(X;\mathbb{Z})$. Let $c \in C_1(X;\mathbb{Z})$ be a solution to the OBCP for b.

Claim: c corresponds to an embedded simplicial path.



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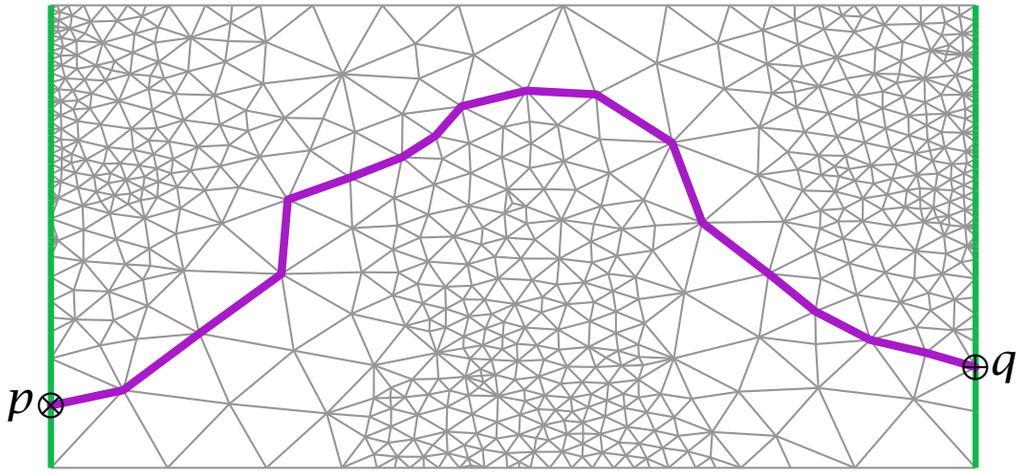

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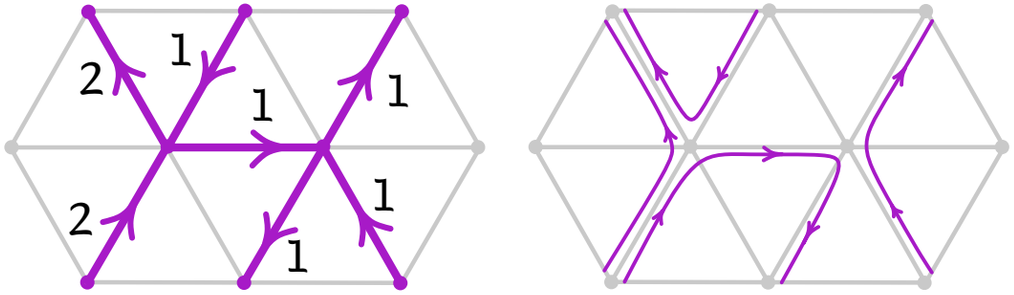

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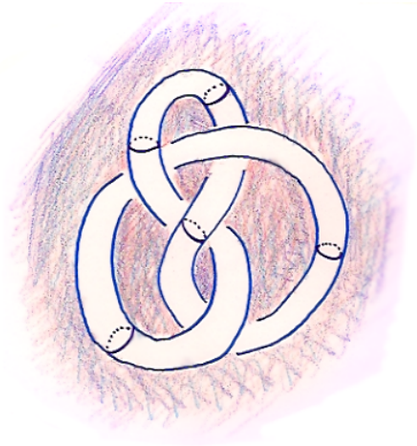

2. Introduce a relative version of the Optimal Bounding Chain Problem.

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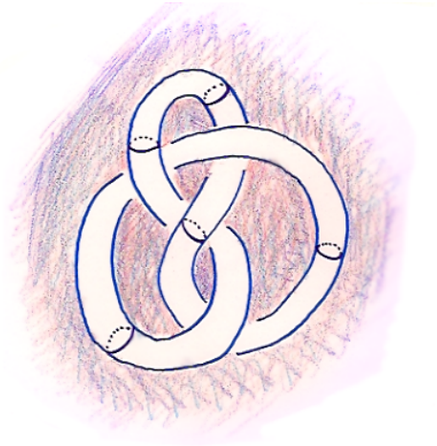

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