Counting essential surfaces in 3-manifolds

Nathan M. Dunfield
University of Illinois

joint with
Stavros Garoufalidis
Hyam Rubinstein

Slides posted at:
http://dunfield.info/slides/CMO2021.pdf

Throughout: M^3 is a cpt orient irreducible with every closed $F^2 \subset M$ orient (e.g. $H_2(M; \mathbb{F}_2) = 0$).

Closed conn embedded $F^2 \subset M$ is *incompressible* when $F \neq S^2$ and $\pi_1 F \to \pi_1 M$ is injective; if F is also not parallel into ∂M, it is *essential*.

Goal: Count (closed) essential surfaces in M, up to isotopy.

T^3: all essential surfaces are tori, infinitely many.

$|\pi_1 M| < \infty$: no essential surfaces.

[Hatcher-Thurston 1985] 2-bridge knot exterior has no ess. surfaces.

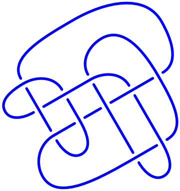
M^3 is *atoroidal* when there are no ess. tori. For atoroidal M, this is always finite:

$$a_M(g) = \#\left\{\text{genus } g \text{ ess. surf, mod iso}\right\}$$

M^3 is *atoroidal* when there are no ess. tori. For atoroidal M, this is always finite:

$$a_M(g) = \#\{\text{genus } g \text{ ess. surf, mod iso}\}$$

For the exterior M of $11n34$:



g	a_M	g	a_M	g	a_M
1	0	7	87	13	602
2	6	8	208	14	1,168
3	9	9	220	15	1,039
4	24	10	366	16	1,498
5	37	11	386	17	1,564
6	86	12	722	18	2,514
					...
				50	56,892
				100	444,038

$$a_M(g) = \#\left\{\text{genus } g \text{ ess. surf, mod iso}\right\}$$

$$b_M(-n) = \#\left\{\begin{array}{c}\text{ess. surf with } \chi = -n \\ \text{mod isotopy}\end{array}\right\}$$

For $M = E_{11n34}$, we show

$$b_M(-2n) = \frac{2}{3}n^3 + \frac{9}{4}n^2 + \frac{7}{3}n + \frac{7 + (-1)^n}{8}$$

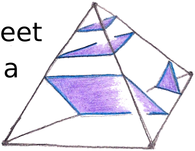
Thm [DGR] For atoroidal M^3, the generating function

$$\sum_{n=1}^{\infty} b_M(-2n)x^n = \frac{P(x)}{Q(x)}$$

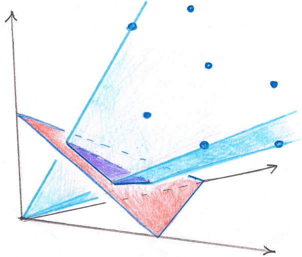
where $P, Q \in \mathbb{Q}[x]$ and Q is a product of cyclotomics.

Algorithm [DGR] Can find P, Q, and isotopy reps for fixed χ.

Normal surfaces meet each tetrahedra in a standard way:



and correspond to lattice points in a finite polyhedral cone P_T in $\mathbb{R}^{7t}$ where $t = \#T$:



Good: Any essential F can be isotoped to be normal.

Bad: Resulting normal surface is far from unique.

weight: $\mathrm{wt}(F) = \#(F \cap T^1)$

lw-surface: an essential normal surface that is least weight in its isotopy class.

[Tollefson 90s, Oertel 80s] Every lw-surface lies on a lw-face $C \subset P_T$, one where **every** lattice point in C is a lw-surface. Isotopies between lw-surfaces can be understood.

[Ehrhart 60s] Counts of lattice points in rational polyhedra are quasipolynomial.

Thm [DGR] For atoroidal M^3, the count $b_M(-2n)$ is quasipolynomial.

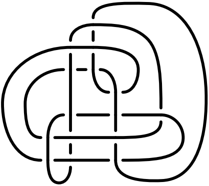
Moral: Ess. surf. are lattice points in the space $\mathcal{ML}(M)$ of measured laminations [Hatcher '90s].

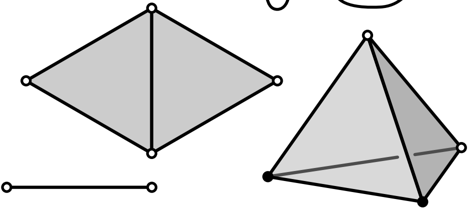
Cor [DGR] The number of ess. surfaces of $\chi = -2n$ grows like n^{d-1} where $d = \dim(\mathcal{ML}(M))$.

[Kahn-Markovic 2012] For M^3 closed hyperbolic, the number of **immersed** essential genus g surfaces grows like g^{2g}.

Computed $\mathcal{LW}_T = \bigcup \left\{ C \text{ is a lw-face} \right\}$ for 59K manifolds. Some 4K with $\dim\left(\mathcal{LW}_T\right) > 1$ giving 88 distinct B_M.

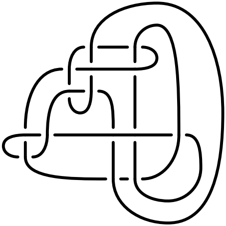
$K15n51747$:

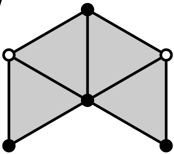


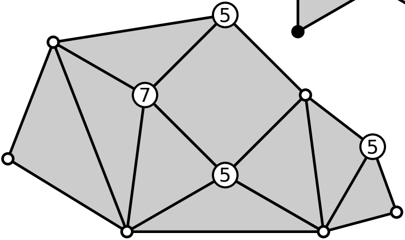


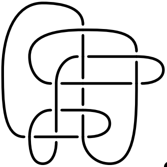
$$\frac{-3x^7+3x^6+9x^5-9x^4-9x^3+9x^2+2x}{(x-1)^4(x+1)^3}$$

$$K15n18579\colon\ B_M(x) = \frac{-2x^6+5x^4-4x^3-15x^2-4x}{(x-1)^3(x+1)^3}$$

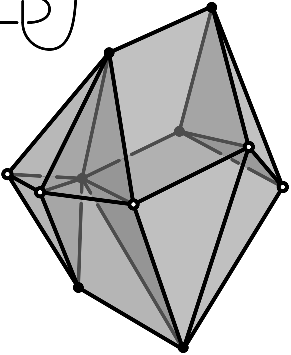








*K*11*n*34



$$B_M(x) = \frac{-x^5 + 3x^4 - 2x^3 + 2x^2 + 6x}{(x+1)(x-1)^4}$$

For $K13n3838$, $\mathcal{LW}_T$ is conn. with 44 maximal faces, all of dim 5, each with 5–9 vertex rays cor. to 48 distinct surfaces of genus 2–5. Here $b_M(-2n)$ is:

$$\frac{7}{12}n^4 + 3n^3 + \frac{14}{3}n^2 + 3n + \frac{7+(-1)^n}{8}$$

and $a_M(g)$ starts 12, 34, 110, 216, 532, 708, 1558, 2018, 3462, 4176, 7314, 7876, 13204, 14256, 20778, 23404, 34820, 34832, 52226,...

What about counting by genus?

$$a_M(g) = \#\left\{\text{genus } g \text{ ess. surf, mod iso}\right\}$$

To compute, need to decide which lattice points correspond to connected surfaces.

For the 4,330 manifolds, see 94 distinct patterns for $a_M(g)$.

The sequence a_M does not determine b_M or conversely.

Even for surfaces, counting connected curves only is very subtle [Mirzakhani]. We only have conjectures.

Conj. For $K13n586$, have $a_M(2) = 2$ and $a_M(g) = \phi(g-1)$ for $g > 2$.

Conj. 54 of our 88 sequences $a_M(g)$ have Möbius transform that is quasipolynomial.

Asymptotics: $\bar{a}_M(g) = \sum_{k \leq g} a_M(k)$

Conj. Either $a_M(g) = 0$ for all large g or there exists $s \in \mathbb{N}$ such that $\lim_{g\to\infty} \bar{a}_M(g)/g^s$ exists and is positive.