



# Practical solutions to hard problems in 3-dimensional topology.

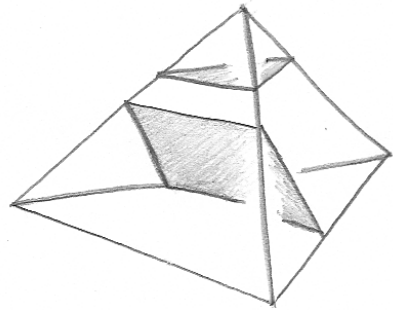
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Fields Institute, November 20, 2009

This talk available at <http://dunfield.info/>  
Math blog: <http://ldtopology.wordpress.com/>

In contrast to higher dimensions, many properties of  $M^3$  are algorithmically computable.

[Haken 1961] Whether a knot in  $S^3$  is unknotted. More generally, find the simplest surface representing a class in  $H_2(M; \mathbb{Z})$ .



[Jaco-Oertel 1984] Whether  $M$  contains an incompressible surface.

[Rubinstein-Thompson 1995] Whether  $M$  is  $S^3$ . Casson showed this allows finding connected sum decompositions.

[Haken-Hemion-Matveev] Whether two Haken 3-manifolds are homeomorphic.

Thurston and Perelman: 3-manifolds have canonical decompositions into geometric pieces modeled on  $\mathbb{E}^3$ ,  $S^3$ ,  $\mathbb{H}^3$ ,  $S^2 \times \mathbb{R}$ ,  $\mathbb{H}^2 \times \mathbb{R}$ , Nil, Sol,  $\widetilde{\text{SL}}_2\mathbb{R}$ .

The work of Perelman, Casson-Manning, Epstein et. al., Hodgson-Weeks, Jaco-Oertel, Haken-Hemion-Matveev, Casson, Rubinstein-Thompson, and others gives

**Thm.** *There is an algorithm to determine if two compact 3-manifolds are homeomorphic.*

Other directions: Heegaard Floer homology, quantum invariants...

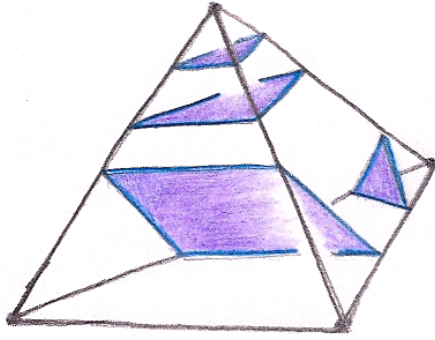
## How hard are these questions?

[Agol-Hass-Thurston 2002] The following is NP-complete:

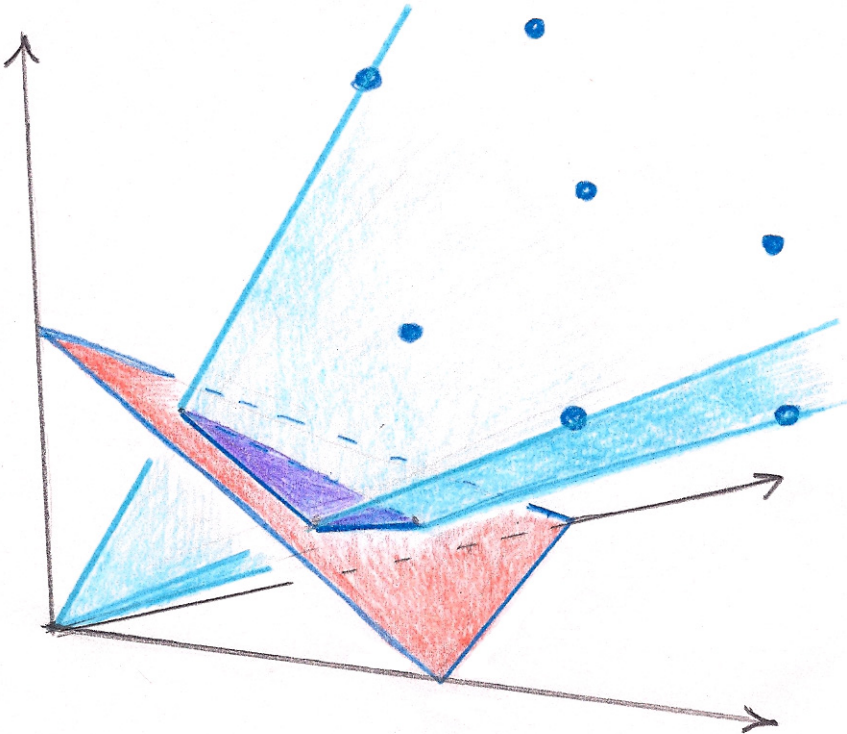
**Q:** Given a manifold  $M$ , a knot  $K$  in  $\mathcal{T}^1$ , and  $g \in \mathbb{N}$ , is there a surface  $\Sigma \subset M$  with boundary  $K$  and genus  $\leq g$ ?

[Casson, Schleimer, Ivanov 2004] Recognizing the 3-sphere is in NP.

Normal surfaces meet each tetrahedra in a standard way:



and correspond to certain lattice points in a finite polyhedral cone in  $\mathbb{R}^{7t}$  where  $t = \#\mathcal{T}$ :



**Meta Thm.** *In an interesting class of surfaces, there is one which is normal. Moreover, one lies on a vertex ray of the cone.*

E.g. The class of minimal genus surfaces whose boundary is a given knot.

Problem: the dimension grows linearly with  $t$ , and moreover there can be exponentially many vertex rays. In practice, limited to  $t < 40$ .

Worse, sometimes have a second step examining each  $M \setminus \Sigma$  and looking for surfaces there, and that new manifold may be much more complicated than  $M$  itself.

**Thm. (Dunfield-Ramakrishnan 2007)** *There is a closed hyperbolic 3-manifold  $M$  of arithmetic type, with an infinite family of finite covers  $\{M_n\}$  of degree  $d_n$ , where the number  $\nu_n$  of fibered faces of the Thurston norm ball of  $M_n$  satisfies*

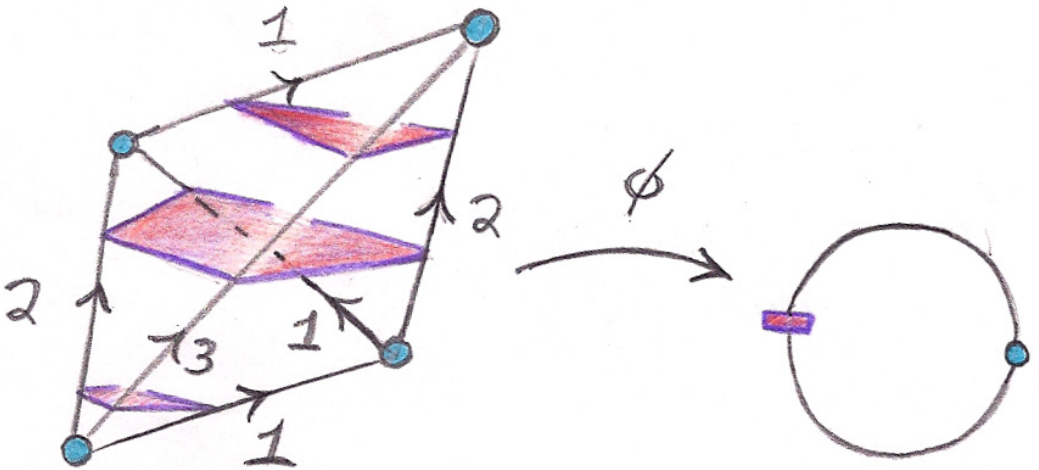
$$\nu_n \geq \exp\left(0.3 \frac{\log d_n}{\log \log d_n}\right) \quad \text{as } d_n \rightarrow \infty.$$

To prove this, we needed to compute the Thurston norm for a manifold with  $\#\mathcal{T} \approx 130$ , and moreover show that it fibers over the circle!



**Practical Trick 1:** Finding the simplest surface representing some  $\phi \in H^1(M; \mathbb{Z}) \cong H_2(M; \mathbb{Z})$ .

Use a triangulation with only one vertex (cf. Casson, Jaco-Rubinstein). The  $\phi$  comes from a unique 1-cocycle, which realizes  $\phi$  as a piecewise affine map  $M \rightarrow S^1$ .



Power of randomization: Trying several different triangulations usually yields the minimal genus surface.

Lower bounds on the genus come from (twisted) Alexander polynomials.

**Practical Trick 2:** Proving that  $N = M \setminus \Sigma$  is  $\Sigma \times I$ .

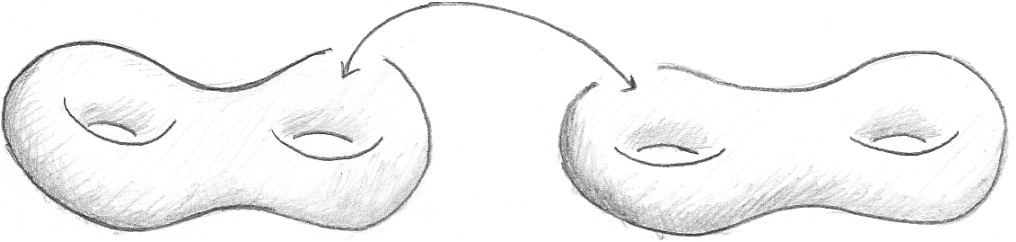
Start with a presentation for  $\pi_1(N)$  coming from a triangulation, and then simplify that using Tietze transformations. With luck (i.e. randomization), one gets a one-relator presentation of a surface group. This gives  $N \cong \Sigma \times I$  by [Stallings 1960].

To see that  $N \not\cong \Sigma \times I$ , try Alexander polynomials.

**Current work:** Can this work for other problems, e.g. finding incompressible surfaces?

## Rank vs. genus (with Helen Wong)

A closed  $M^3$  can always be constructed as



Consider

$\text{rank}(M) = \min$  genus of a Heegaard splitting

$\text{genus}(M) = \min$  size of a gen set of  $\pi_1 M$

Clearly have  $\text{rank}(M) \leq \text{genus}(M)$ .

**Q.** *Does  $\text{rank}(M) = \text{genus}(M)$  for all hyperbolic 3-manifolds?*

[Boileau-Zieschang 1984] There are Seifert fibered spaces with  $\text{rank}(M) \neq \text{genus}(M)$ .

Further graph manifold examples found by Weidmann and Schultens.

## Searching for an example.

Computability in theory for  $M^3$  hyperbolic:

- **Rank:** Yes [Kapovich-Weidmann 2004]
- **Genus:** Unknown, likely yes. Rubinstein and Stocking showed that (many) Heegaard surfaces can be made almost normal, but there are infinitely many candidates surfaces.

[Lackenby 2008] When  $M$  has cusps, can compute the genus by using the right triangulation.

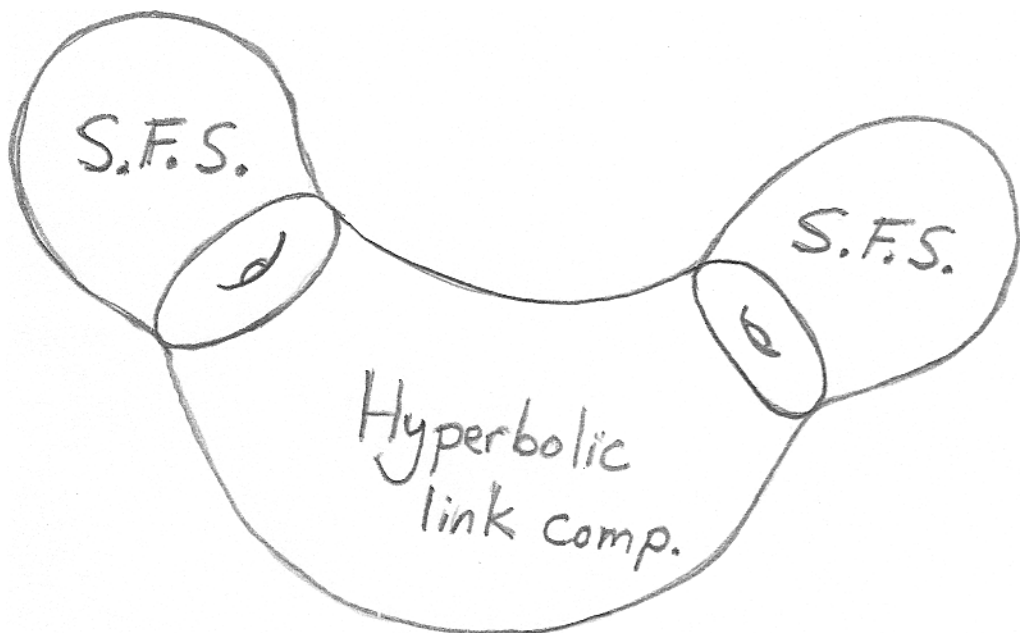
## Computability in practice.

- **Rank:** Occasionally. Can search for smaller generating sets via Todd-Coxeter coset enumeration. Lower bounds are hard to come by, except for the rank of  $H_1(M; \mathbb{Z})$ .
- **Genus:** Sometimes. Start with a presentation of  $\pi_1(M)$  coming from a triangulation, then simplify via Tietze transformations. The result inevitably comes from a Heegaard splitting of  $M$ . Using randomization, can get a good idea of what the genus should be. Lower bounds, other than the rank, are few, e.g. quantum invariants.

Note: Quantum invariants can be used to reprove the examples of Boileau-Zieschang [Wong 2007].

So far, we don't even have any *candidate* hyperbolic examples, even though our methods quickly find many of the known non-hyperbolic examples.

We think we've found a new non-hyperbolic example:



We know that  $\text{rank}(M) = 3$  and *strongly suspect* that  $\text{rank}(M) = 4$ .

# SnapPy

## What is SnapPy?

SnapPy is a user interface to the SnapPea kernel which runs on Mac OS X, Linux, and Windows. SnapPy combines a link editor and 3D-graphics for Dirichlet domains and cusp neighborhoods with a powerful command-line interface based on the [Python](#) programming language. You can see it [in action](#), learn how to [install](#) it, and read the [tutorial](#).



## Contents

- [Screenshots: SnapPy in action](#)
- [Installing and running SnapPy](#)
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- [Using SnapPy's link editor](#)
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- [Development Basics: OS X](#)
- [Development Basics: Windows XP](#)

## Credits

Written by [Marc Culler](#) and [Nathan Dunfield](#). Uses the SnapPea kernel written by [Jeff Weeks](#). Released under the terms of the GNU General Public License.

<http://www.youtube.com/user/NathanDunfield/>



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## SnapPy tutorial, Part I: Basics

```
Hi. It's SnapPy.  
SnapPy is based on the SnapPea kernel, written by Jeff Weeks.  
Type "Manifold?" to get started.  
  
In []: |
```

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