Groups & 3-manifolds: Nathan PS 3   Wednesday, June 7, 2023.

1. Suppose N is a compact orientable 3-manifold with ∂N a nonempty union of tori. Let $M = N \setminus \partial N$ be its interior. Show that for any ideal triangulation $\mathcal{T}$ of M the number of tetrahedra in $\mathcal{T}$ is equal to the number of edges.

2. Prove that any ideal tetrahedra in $\mathbb{H}^3$ has volume less than 1000. (In fact, the volume is less than 1.01494160640965362502121.)

3. Every cusped hyperbolic 3-manifold has a "canonical cell decomposition" which is typically a triangulation and hence often referred to as the "canonical triangulation".

   (a) Figure out the method in SnapPy for replacing a triangulation with the canonical one. Use this to find an example where the the canonical triangulation does *not* minimize the number of tetrahedra.

   (b) If you "browse" a manifold in SnapPy, the "Cusp Nbhds" tab shows what you see if you stand infinity in the first cusp of the 3-manifold and look inside. It includes a visualization of both the canonical triangulation and its dual, called the Ford domain. Use the "View options" menu to turn on and off different parts of the visualization and try to understand what it is the picture is telling you.

4. Here's how you get the exterior of a randomly chosen 14-crossing prime knot:

```
knots = HTLinkExteriors(cusps=1, crossings=14)
M = knots.random()
```

   (a) Python uses the `len` function to access the length of any list-like object, so do `len(knots)` to see how many such knots there are.

   (b) Try creating the Dirichlet domain for M at the command line. Most of the time you will get an error message saying that this failed! (If not, pick a different example which does fail for the rest of this problem ;-).

   (c) By default, the hyperbolic structure on M is computed using standard double-precision floating-point numbers (about 15 decimal digits). It turns out this isn't enough to find the Dirichlet domain for a manifold this complicated. Use the `high_precision` method of M to upgrade it to a `ManifoldHP` called H.

   (d) Compute the volumes of M and H. Are the answers consistent with the hyperbolic structure on H being computed to quad-double precision?

   (e) Try computing the Dirichlet domain D using H, which will most likely succeed though it typically takes a few seconds.

   (f) Interrogate D to get a pretty picture and find out how many faces and vertices D has.

5. Programming and searching.

   (a) Find all manifolds in the `OrientableClosedCensus` which have a 2- or 3-fold covering space with $b_1 > 0$.

(b) The smallest volume manifold in the `OrientableClosedCensus` with $b_1 > 0$ is $m160(3,1)$ which has volume ≈ 3.1663333212. Find a closed manifold with $b_1 > 0$ with smaller volume by searching through 0 surgeries on the knots in `HTLinkExteriors`.

(c) Express the first manifold M that you found in (b) as a Dehn filling on a 1-cusped manifold in the `OrientableCuspedCensus`.

(d) The reason that the closed manifold M in part (c) is not in the `OrientableClosedCensus` is that its shortest geodesic has length smaller than the cutoff that was chosen by Hodgson and Weeks when they created this census. Determine this cutoff by finding length of the shortest geodesic for the all the manifolds in the `OrientableClosedCensus`.

6. Recall that two 3-manifolds are commensurable if they have a common finite cover.

   (a) In the `OrientableClosedCensus`, find several pairs of 1-cusped manifolds that appear to have the same volume. (You don't have to find all such pairs, though that can certainly be done.)

   (b) For each pair, increase your confidence that the volumes are the same by using the `ManifoldHP` type, which works with quad-double precision.

   (c) For each pair, try to either (a) find a common finite cover, or (b) show they are incommensurable by looking at the cusp density.