Groups around 3-manifolds: Nathan PS 2 Monday, June 5, 2023.

For lecture notes and references, see: `http://dunfield.info/crm2023`

1. The Poincaré Conjecture posits that any closed 3-manifold M with $\pi_1(M)$ trivial is homeomorphic to the 3-sphere S^3. Show that this follows from the Geometrization Theorem.

2. A quick way to see that S^3 is Seifert fibered is to view it as the unit sphere in $\mathbb{C}^2$:

$$S^3 = \left\{ (u, v) \in \mathbb{C}^2 \;\middle|\; |u|^2 + |v|^2 = 1 \right\}$$

   and consider the diagonal action of $S^1 = \{z \in \mathbb{C} \mid |z| = 1\}$ on $\mathbb{C}^2$: $z \cdot (u, v) \mapsto (zu, zv)$. This action is free and leaves S^3 invariant. Its orbits give a foliation of S^3 by circles; the quotient space is S^2 and the map $S^3 \to S^2$ the usual Hopf fibration. Use the same technique to give a Seifert fibered structure to any lens space $L(p, q)$. What is the quotient in that case?

3. Suppose M is a closed hyperbolic n-manifold.

   (a) Prove that every $\gamma \neq 1$ in $\pi_1(M)$ is homotopic to a unique closed geodesic.

   (b) Prove that for every $L \geq 0$ there are finitely many closed geodesics of length at most L.

   (c) Prove that the isometry group of M is finite.

4. Mostow rigidity says that if M and N are closed hyperbolic n-manifolds of finite-volume and $n \geq 3$, then any group isomorphism $\phi\colon \pi_1 M \to \pi_1 N$ is induced by an isometry $M \to N$. In particular, M and N are diffeomorphic. (This is not true for all 3-manifolds: there are lens spaces with the same π_1 that are not even homotopy equivalent.) Use Mostow to prove that the outer automorphism group $\mathrm{Out}(\pi_1(M))$ is finite for any finite-volume hyperbolic n-manifold of dimension ≥ 3. (For the $n = 2$ case of a closed hyperbolic surface, the group $\mathrm{Out}(\pi_1(M))$ is the mapping class group, which is both infinite and highly interesting.)

5. The remaining problems all involve practical computation with hyperbolic structures, so the first step is to download and install **SnapPy** from `http://snappy.computop.org`.

6. (a) Load the manifold $v1234$ and name it V.

   (b) Use the browser to find the volume, Dirichlet domain, and symmetry group of V.

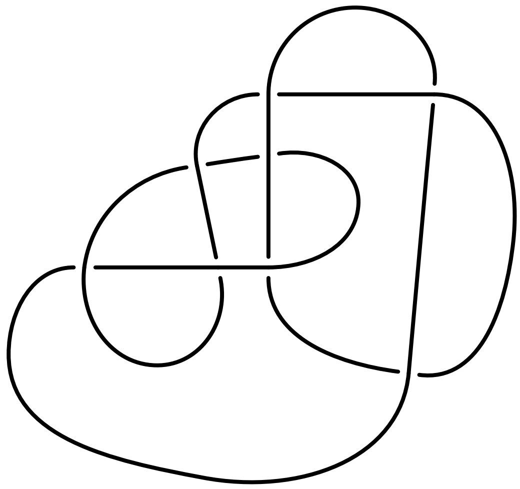
   (c) Like any manifold in SnapPy, the object V is really a particular *triangulation* of this hyperbolic manifold. Back at the command line, determine the number of tetrahedra in the triangulation V. Hint: Use tab completion by typing `V.<tab-key>`.

   (d) The manifold V has one cusp. Back the browser, do Dehn filling along the meridian curve. What closed manifold do you get?

7.

(a) Use SnapPy to find the name in the Rolfsen table for the link shown at right.

(b) Is the projection at right the same as the one that's stored in SnapPy?



8. In my lecture, I mostly focused on manifolds with cusps, but SnapPy also works with closed manifolds. In particular, it comes with the Hodgson-Weeks census of small-volume closed hyperbolic 3-manifolds, which is called `OrientableClosedCensus`.

(a) Use the "?" operator to find out more about `OrientableClosedCensus`; in particular, how many manifolds are in it?

Also, interrogate the orientable *cusped* census to get ideas on how to select various types of manifolds for the later parts of this question.

(b) Closed manifold in SnapPy are represented as Dehn fillings on cusped manifolds. You can do Dehn filling in the browser, via the `dehn_fill` method, or as part of the specification that you given to `Manifold`. For example, typing `A = Manifold('4_1(1,2)')` gives the closed 3-manifold which is $\frac{1}{2}$-Dehn surgery on the figure 8 knot. Use the method `is_isometric_to` to show that A is the sixth manifold the `OrientableClosedCensus`. Warning: In Python, lists are numbered starting from 0 rather than 1.

(c) Find the unique manifold M in the original `OrientableClosedCensus` whose volume is between 3.0 and 3.1 and whose first homology is $\mathbb{Z}/3\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z}$.

(d) Find a description of M as Dehn surgery on a 3-component link in S^3. Hint: Unfill the cusp in the default description of M and then drill out the shortest geodesic twice.