Groups around 3-manifolds: Nathan PS 1 Monday, June 5, 2023.

For lecture notes and references, see: http://dunfield.info/crm2023

1. A 3-manifold is *irreducible* if every smoothly embedded 2-sphere bounds a 3-ball. For example, a basic topological fact is that $\mathbb{R}^3$ and S^3 are irreducible (this was first shown by Alexander and is true in any dimension, where it is usually referred to as the Schoenflies Theorem).

   (a) Show the only closed orientable 3-manifold that is prime but not irreducible is $S^2 \times S^1$.

   (b) Prove that if $\widetilde{M} \to M$ is a covering map and $\widetilde{M}$ is irreducible then so is M.

   (c) Use (b) to show that T^3 and all the lens spaces $L(p,q)$ are irreducible.

2. Use the Sphere Theorem to prove that if M^3 is a closed orientable irreducible 3-manifold with $G = \pi_1 M$ infinite, then M is an Eilenberg-MacLane space $K(G,1)$. That is, prove that the universal cover of M is contractible.

3. Prove that the *abelian* groups that appear as fundamental groups of closed orientable 3-manifolds are exactly the cyclic groups of all orders (including $\mathbb{Z}$) and $\mathbb{Z}^3$. Hint: Use Problem 2 and note you do *not* need geometrization for this.

4. For an orientable closed surface S with some fixed Riemannian metric, consider the circle bundle $UT(S) = \{v \in T_* S \mid \|v\| = 1\}$ of unit-length tangent vectors. (The topology of $UT(S)$ does not depend on the metric.) Show that $UT(S)$ admits a Riemannian metric modelled on one of the eight Thurston geometries. Hint: Which geometry to pick depends on S!

5. Prove that $T^3 \# T^3$ cannot be given a geometric structure modelled on one of the eight Thurston geometries.

   More precisely, suppose that X is one of $\mathbb{E}^3$, S^3, $\mathbb{H}^3$ $S^2 \times \mathbb{R}$, $\mathbb{H}^2 \times \mathbb{R}$, Nil, Sol, $\widetilde{\mathrm{SL}_2 R}$. Prove that $T^3 \# T^3$ does not have a Riemannian metric such that every points $p \in T^3 \# T^3$ has an open ball isometric to one in X. Equivalently, show that $T^3 \# T^3$ is not homeomorphic to $\Gamma \backslash X$, where Γ is a subgroup of $\mathrm{Isom}^+(X)$ that acts freely and properly discontinuously on X.