Groups around 3-manifolds

Lecture 2

Convention: All manifolds will be orientable

Homeomorphism Problem: Given two closed n-mflds M and N (say as simpl. complexes), are they homeomorphic? Is this decidable? [i.e. $\exists$? a computer algorithm]

[Context: Whether two finite presentations give isomorphic gps is undecidable.]

$n=1$: Yes — return "yes".

$n=2$: Yes — return $\chi(M) = \chi(N)$.

$n \geq 4$: No — [Markov 1958]

$n=3$: Yes — Cor. of Geometrization; see [Kuperberg 2017]

Outline: 1: Find the geometric (JSJ) decomposition. (Normal surfaces)

2: Non hyp. pieces are classified.

3. For hyp. pieces need to check for an isometry (by Mostow Rigidity).

An M^3 is Seifert fibered if it has a foliation by circles.

<u>Ex</u>: $F^2 \times S^1$, $UT($$)$, S^3 (Hopf fib.), ...

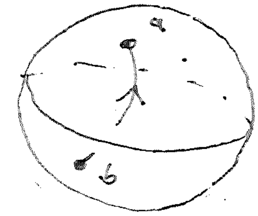
<u>Note</u>: A Seifert fib. space cannot have a hyp str as up to finite covers

$$1 \to \underset{\text{central}}{\mathbb{Z}} \to \pi_1 M^3 \to \pi_1 F^2 \to 1$$

<u>Contrast</u>: $\pi_1(\text{hyp}) \leq \text{Isom}^+(\mathbb{H}^3) = \underbrace{PSL_2\mathbb{C}}_{\text{no center}}$

<u>Fact</u>: M^3 closed hyp, then $\mathbb{Z}^2 \not\leq \pi_1 M$.

<u>Pf</u>: Every nontriv elt of $\pi_1 M$ is hyperbolic has an axis an fixes two pts on $\partial_\infty \mathbb{H}^3$.



A pair of commuting hyp must have the same fixed pts.

$\underset{\text{discrete}}{\Longrightarrow}$ part of a cyclic subgp. $\square$

<u>Contrast</u>: Finite-vol hyp 3-mflds have cusps.

$\rightsquigarrow \mathbb{Z}^2 \leq \pi_1 M$.

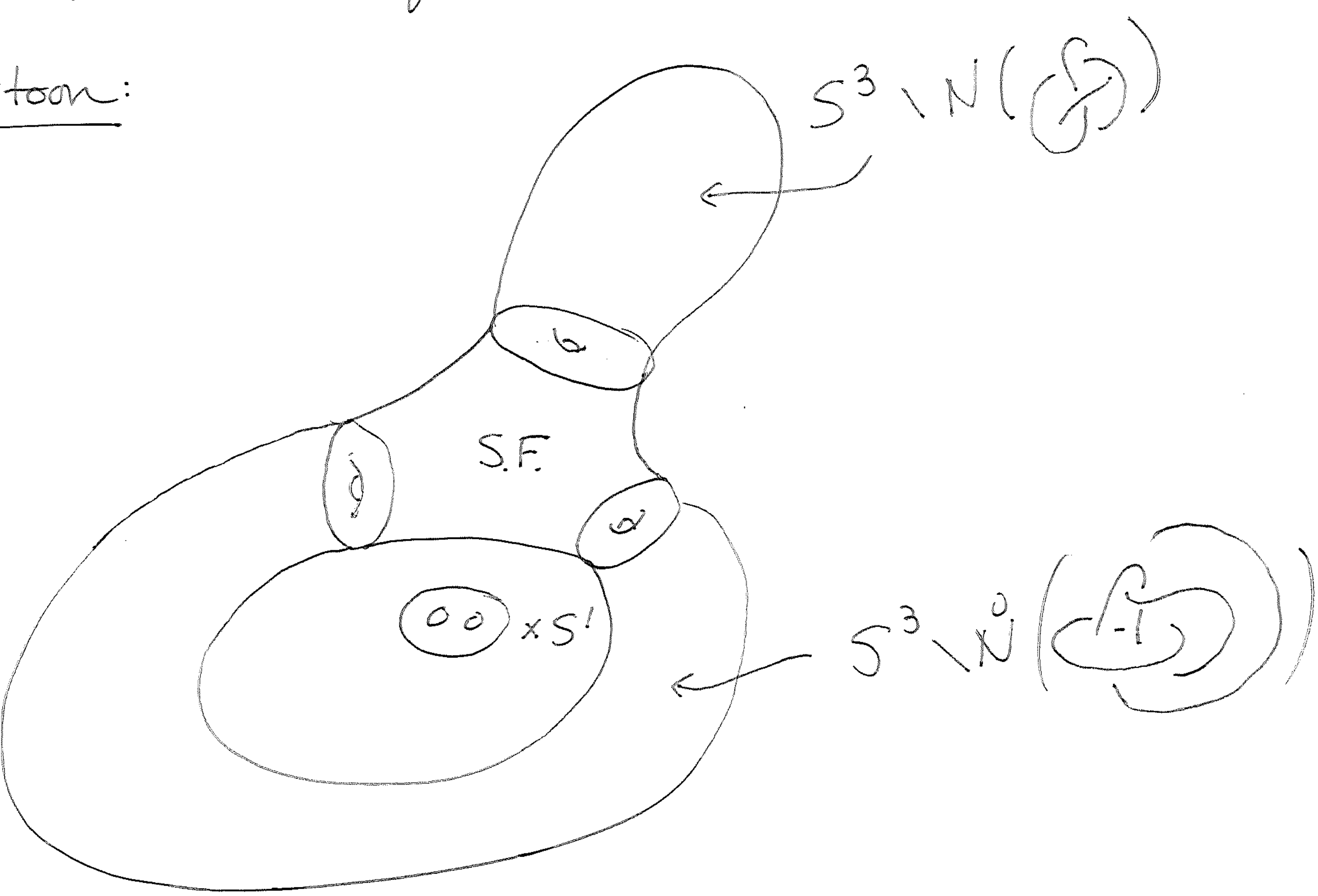
Hardest part of Geometrization: (M $\overset{\text{closed}}{\text{irred}}$, $\pi_1 M$ infinite, no $\mathbb{Z}^2$ in $\pi_1 M$) $\Longrightarrow$ M hyperbolic.

A surface $(F, \partial F) \hookrightarrow (M, \partial M)$ is <u>incompressible</u> when $F \neq$ [sphere] or [disk] and $\pi_1 F \to \pi_1 M$ is 1-1.

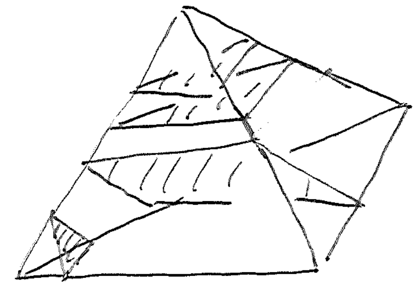
<u>Def</u>: M^3 is atoroidal when every incomp. torus is parallel into ∂M. <u>Hyp. mflds are atoroidal</u>

<u>Jaco-Shalen-Johannson</u>: Any compact irreducible M^3 has a collection T of disjoint incomp. tori such that each comp of $M \setminus T$ is either atoroidal or Seifert fibered; a minimal such T is unique up to isotopy.

<u>Cartoon</u>:

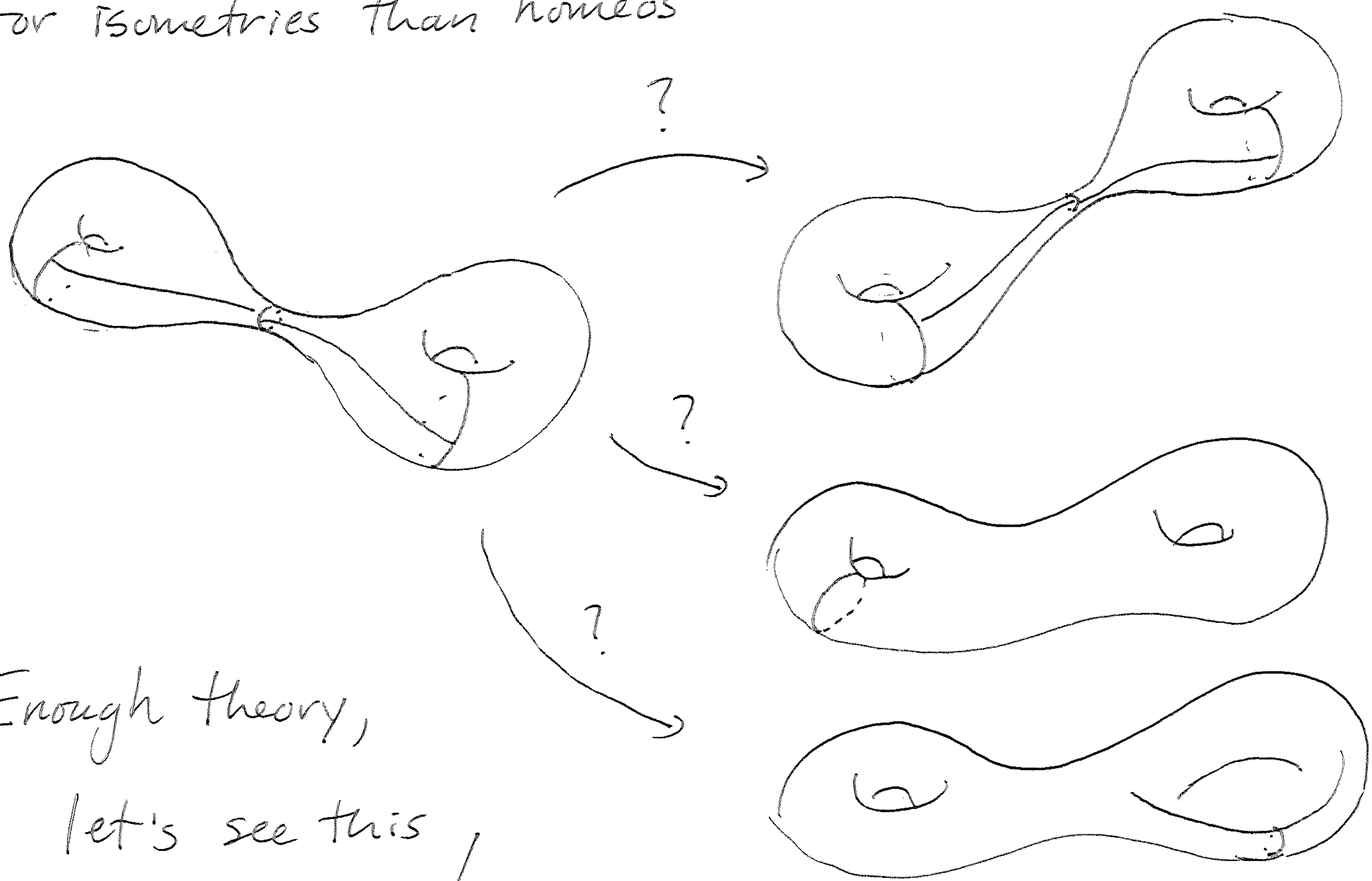


Normal Surfaces [Haken 1960s...]



incomp can be isotoped to normal.

Role of Geometry: Much easier to check for isometries than homeos



Enough theory, let's see this in practice!