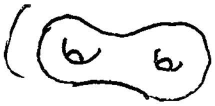
Groups Around 3-manifolds: Summer school CRM Montréal 2023.

Intro talk: Basic topology and geometry of 3-mflds.

2D: Up to finite-index $\pi_1(\text{cpt surface}) = 1, \mathbb{Z}, \mathbb{Z}^2, F_2$, or $\pi_1($  $)$

4D: Any finitely-presented G is π_1 of a smooth closed W^4. [3D inbetween...]

Cor: M^3 cpt. Then $G = \pi_1 M$ is residually-finite: $\bigcap \{ H \leq G \mid [G:H] < \infty \}$ [Cor of what?...]

Fact: Any mfld of dim 2 or 3 has a unique smooth structure. [no cptness needed; same true for P.L.]

Contrast: $\mathbb{R}^4$ has uncountably many smooth strs.

Conventions: (1) All mflds/maps are smooth.
(2) All 3-manifolds are oriented and typically closed.

Connect sum:

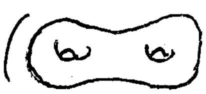
A B A # B

Prime: $M^3 = A \# B \Rightarrow$ one of $A, B \cong S^3$.

[Kneser-Milnor] Any closed M^3 is $A_1 \# \cdots \# A_k$ with A_k prime. The A_i are unique up to perm.

Note: $\pi_1(A \# B) \cong (\pi_1 A) * (\pi_1 B)$

[Stallings] if $\pi_1 M^3 \cong \Gamma_1 * \Gamma_2$ then $M = A_1 \# A_2$ with $\pi_1 A_i \cong \Gamma_i$.

Prime 3-mflds: S^3, $L(p,q)$, $T^3 = S^1 \times S^1 \times S^1$, $\Sigma_g \times S^1$ ($g \geq 1$), $UT($$)$, ...

These are all irreducible: every embedded S^2 bounds a ball. $S^2 \times S^1$ is the only clsd prime mfld that is not irred. Pfs: HW!

Theme in 2D and 3D: promoting homotopy to homeo/isotopy.

Sphere Thm [Papakyriakopoulos] If $\pi_2 M^3 \neq 0$ there exists an embedded S^2 thats $\neq 0$ in $\pi_2 M$.

Cor: If M^3 is clsd irred with $|\pi_1 M| = \infty$ then M is a $K(\pi_1 M, 1)$.

Pf: HW.

Role of Geometry:

$n=2$: Any closed surface has a const. curv. metric.

$n \geq 4$: Homogenous geometry is "rare."

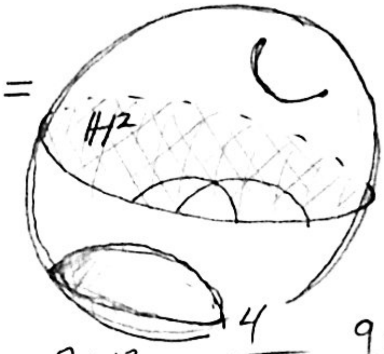
$n=3$: Some have no const. curve metrics: $S^2 \times S^1$
(Univ. cover not homeo to S^3, $\mathbb{E}^3$, $\mathbb{H}^3$)

Geometrization Thm: [Thurston, Perelman, ...]
Any irred closed M^3 can be decomposed② along incompressible①  to pieces w/ metrics modelled on one of: S^3, $\mathbb{E}^3$, $\mathbb{H}^3$, $S^2 \times \mathbb{R}$, $\mathbb{H}^2 \times \mathbb{R}$, Nil, Solv, $\widetilde{SL_2 \mathbb{R}}$.

① **incompressible**: embedded $F^2 \subseteq M^3$ where $\pi_1 F \hookrightarrow \pi_1 M$.

② **decomposed**: Jaco - Shalen — Johannson

$\{|x|<1\} =$



Most important/common geometry: $\mathbb{H}^3 =$

[Mflds w/ other geoms are classified...]

$$g_{\mathbb{H}^3} = \frac{4}{(1-r^2)^2} g_{\mathbb{E}^3}$$

A hyperbolic structure on M^3 is a Riem. metric of const curve -1. Equivalently,

$$M = \Gamma \backslash \mathbb{H}^3 \qquad \Gamma = \pi_1 M \le \mathrm{Isom}^+(\mathbb{H}^3)$$

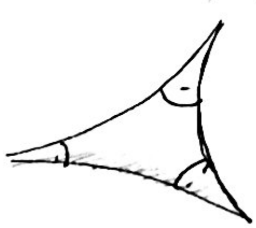
discrete, torsion free

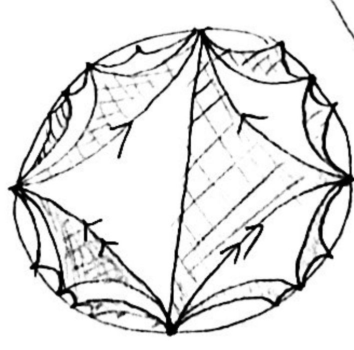
$$\mathrm{Isom}^+(\mathbb{H}^3) = \text{Möb}^+(\hat{\mathbb{C}}) = PSL_2\mathbb{C}$$

Motivating example:

$$\Lambda = \left\langle \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} \right\rangle$$

$\Lambda \backslash \mathbb{H}^2 =$





Not compact but area $= 2\pi$

$$\Gamma = \left\langle \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 2-i & 1 \\ 2i & i \end{pmatrix}, \begin{pmatrix} 3 & 2i \\ 2i & -1 \end{pmatrix} \right\rangle$$

$$M = {}_{\Gamma}\backslash^{\mathbb{H}^3} = S^3 \setminus$$  Borromean Rings $= 6^3_2$

Volume $\approx 7.327724753\ldots$

<u>Mostow Rigidity</u>: Suppose M and N are hyperbolic n-manifolds of finite volume where $n \geq 3$. If $\pi_1 M \cong \pi_1 N$ then M and N are <u>isometric</u>.

<u>Cor</u>: Any geometric invariant of a hyp. n-mfld for $n \geq 3$ is a topological invariant.

Thurston's Mantra: "Topology = Geometry" in dim 3.