Math 277: Exercises for weeks 4-

Last Revised: October 30, 2000

Note: These are just informal and not to be turned in. The difficulty varies widely. These problems will be updated regularly and can be downloaded from the course web page.

The 8-geometries

1. For further problems, see Chapters 3 and 4 of Thurston, which has many exercises.
2. Show that every isometry of Nil and $\widetilde{\mathrm{SL}_2\mathbb{R}}$ is orientation preserving.
3. Show that the only closed 3-manifolds which have $S^2 \times \mathbb{R}$ geometries are $S^2 \times S^1$, $\mathbb{R}\,\mathrm{P}^3 \# \mathbb{R}\,\mathrm{P}^3$, $S^2 \tilde{\times} S^1$, and $\mathbb{R}\,\mathrm{P}^2 \times S^1$ (See Thurston Ex. 4.7.1 for an outline of the proof).
4. Understand Nilgeometry: Thurston Ex. 3.5.8, 3.8.7.
5. Let ϕ be a mapping class of the torus T^2. Show that the mapping torus of ϕ, the 3-manifold $M_\phi = T^2 \times I/\{(x,1) \sim (\phi(x),0)\}$ has a geometric structure, either $\mathbb{E}^3$, Nil, or Sol. (See Thurston Ex. 3.8.10 for more).
6. Prove that the only non-irreducible closed 3-manifold which has a geometric structure is $\mathbb{R}\,\mathrm{P}^3 \# \mathbb{R}\,\mathrm{P}^3$.
7. Calculate the sectional curvatures for $\mathbb{H}^2 \times \mathbb{R}$ and $\mathrm{SL}_2\mathbb{R}$. Are they the same?
8. Show that any closed 3-manifold with a Sol geometry fibers over a 1-dimensional orbifold with fiber a torus and/or Klein bottle.

Orbifolds

1. Enumerate all of the 1-dimensional orbifolds.
2. Find all the 2-orifolds with Euler characteristic 0, and thus enumerate the 17 Euclidean wallpaper groups.
3. Suppose that X is a smooth n-dimensional orbifold. Show that we can take the charts on X to be of the form U/Γ where U is an open set in $\mathbb{R}^n$ and Γ is a finite subgroup of $\mathrm{O}(n)$.
4. By the proceeding problem, a nbhd of a point in an n-dimensional orbifold is homeomorphic to the cone on a $(n-1)$-dimensional spherical orbifold. Use this to show that the underlying space of an orientable 3-orbifold is a manifold, and that the underlying space of non-orientable 3-orbifold is not always a manifold.
5. Let Y be an compact orbifold (w/o boundary) which has an (X,G)-geometric structure where (X,G) is a model geometry. Show that the universal cover of Y, $\tilde{Y}$ is a manifold, and that $Y = X/\Gamma$ for some discrete subgroup $\Gamma \subset G$. (Note, this isn't so easy). This implies that the bad 2-dimensional orbifolds do not have metrics of constant curvature +1.

Seifert Fibered Spaces

1. Show that the definition of the Euler number of a Seifert Fibered space is independent of all the choices involved, and that the two definitions that I gave agree.

2. Let M be Seifert fibered. Show that M contains an incompressible torus unless the Seifert orbifold is an S^2 with 3 or fewer cone points.

3. Let M be Seifert fibered, and let S be an incompressible surface in M. Show that S can be isotoped to be either *vertical* (a union of fibers) or *horizontal* (transverse to the fibers). (This is in Hatcher's notes if you get stuck).

4. Prove that the Bieskorn Spheres:

   $$B(p,q,r) = \{(x,y,z) \in \mathbb{C}^3 | x^p + y^q + z^r = 0\} \cap S^5_\epsilon$$

   are Seifert fibered with three exceptional fibers. Show that the orbifold is a sphere with 3 cone points of orders (p,q,r).

5. Show that S^3 has a Seifert fibration over every 2-orbifold with underlying space the S^2. Is this true for lens spaces as well?

6. One can also consider Seifert fibered manifolds with boundary. Show that if M is an orientable S.F. manifold with boundary then the boundary is a union of tori.

7. A good example of a S.F. manifold with boundary is the exterior M of the trefoil knot in S^3, that is $M = S^3 \setminus$ open nbhd of knot:

   Show that M is S.F. with base orbifold a disc with cone points of orders 2 and 3 (Note: this is an orbifold with boundary, like a manifold with boundary. The boundary of the disc is not mirrored.) Hint: Take a regular nbhd of the Möbius band shown to get an annulus in M which splits M into two solid tori. Give each of these tori a S.F. structure. Also, show that $\pi_1(M)$ is the free product with amalamagtion $\mathbb{Z} *_{\mathbb{Z}} \mathbb{Z}$ where the center $\mathbb{Z}$ includes as $2\mathbb{Z}$ and $3\mathbb{Z}$ respectively.

   If we think of the complement N of the trefoil ($= S^3 \setminus$ the knot), then this is a S.F. space with non-compact base orbifold a sphere with two cone points and also a point removed. This is a hyperbolic orbifold B of finite volume, namely $\mathbb{H}^2/\mathrm{PSL}_2\mathbb{Z}$. Understand the map from π_N to $pi_1(B)$ (Hint: Recall that $\mathrm{PSL}_2\mathbb{Z} = \mathbb{Z}_2 * \mathbb{Z}_3$.)

   Does N have a finite volume $\mathbb{H}^2 \times \mathbb{R}$ structure or a $\widetilde{\mathrm{SL}_2\mathbb{R}}$ structure?