Math 20 -- Linear discrete dynamical systems, Markov matrices and Leslie matrices

1) Find a 2 by 2 matrix **A** for which (3,1) and (1,2) are eigenvectors, with eigenvalues 5 and 10, respectively.

2) Consider the matrix $\mathbf{A} = \begin{pmatrix} -15 & 8 \\ -24 & 13 \end{pmatrix}$:

   a) Find all the eigenvalues and eigenvectors of **A**.
   b) Compute $\mathbf{A}^7$.
   c) Estimate the size of the lower left entry of $\mathbf{A}^{1995}$.

3) Let $\mathbf{A} = \begin{bmatrix} -4 & 26 & -6 \\ 0 & 0 & 0 \\ 3 & -21 & 5 \end{bmatrix}$.

   a) Find the characteristic polynomial $f_{\mathbf{A}}(\lambda)$ of **A**.
   b) Find the eigenvalues and eigenvectors of **A**.
   c) Using the results of b) (and without using a calculator to compute $\mathbf{A}^{60}$), find $\mathbf{A}^{60}\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$.

4) a) Find the eigenvalues and eigenvectors of the matrix $\mathbf{A} = \begin{bmatrix} -7 & 1 & 7 \\ -9 & 1 & 9 \\ -5 & 1 & 5 \end{bmatrix}$.

   b) Calculate $\mathbf{A}^7$. You can verify with a calculator or by brute force, but solution should use the above results.

5) A certain rodent has a reproductive life of 3 years. Each female will, on average, produce its first 3 female offspring during its second year and 5 more female offspring during its third year. Unfortunately, only 25% of these rodents survive their first year and of those who survive, only 20% will make it to the third year. This population will eventually stabilize. If we were to start off with 100, 200, and 400 female rodents and sufficient males to assist the process, how will the eventual population be distributed among the three age groups, in terms of percentages? If you solve this problem via calculator, you should also give the actual populations of 1, 2, and 3 year old female rodents and the total female population.

6) A bird species has a maximum life span of 3 years. On average, each pair of birds in their first year will produce two offspring. A typical sample of 8 birds in their 2nd year will produce a total of 15 offspring. After their second year, birds produce no more offspring. Only 40% of birds in their first year will survive to their second year, and only 30% of birds in their second year survive to their third year. Survival rates do not depend on gender.

   Describe how this bird population evolves over time. In particular, describe whether the population remains stable, decreases, or increases and at what rate. Furthermore, if possible describe the relative proportions of each age group after a number of years have passed.

7) There is a relatively unknown world of vampires and werewolves among us. What you may not know is that werewolves can be transformed into vampires and vampires into werewolves. In fact, each year 30% of vampires become werewolves and 10% of werewolves become vampires. Vampires cannot be killed, but werewolves <u>can</u> die and, in fact, lose 10% of their number each year in this way. The only way into this unknown world is to be bitten by a vampire, thus becoming a vampire. Only one in five vampires manage to successfully create a new vampire in a given year.

   Describe how the population of vampires and werewolves will evolve over time. In particular, determine whether their numbers will stabilize, decrease, or increase and at what rate, and what percentages of this unknown world will be comprised by vampires and by werewolves.

8) A certain fish population has two age classes, and a Leslie matrix $\mathbf{L} = \begin{pmatrix} 1 & 3/2 \\ 1/2 & 0 \end{pmatrix}$.

   a) If the population vector at the start of the n-th growth period is $\mathbf{p}_n$, and if $\mathbf{p}_{n+1} = \mathbf{L}\mathbf{p}_n$, what happens to $\mathbf{p}_n$ as $n \to \infty$.
   b) Now suppose that, in addition, at the end of each growth period, a fraction h_1 of the first age class is caught and killed, and h_2 of the second. Show that $\mathbf{p}_{n+1} = (\mathbf{I}-\mathbf{H})\mathbf{L}\mathbf{p}_n$ where **I** is the identity matrix and $\mathbf{H} = \begin{pmatrix} h_1 & 0 \\ 0 & h_2 \end{pmatrix}$.
   c) Now suppose that $h_1 = h_2 = h$. Is there any value of h which will allow the fish population to come to a stable state? If not, explain why not; if so, find the value of h and the stable state.

9) Recently, the Broadway Marketplace opened next to Starbuck's a couple of blocks from here, selling fresh meat and produce and some groceries. For argument's sake, let's say that this store has one competitor, Megamart Inc., and that there is a total customer base of 5,000 people, all of whom have been going to Megamart. Weekly surveys show that Megamart retains 75% of it's customers with the rest going to Broadway. Broadway keeps 60% of its customers from one week to the next, with the rest going for the cheaper prices at the Megamart. Broadway's consultant tells them that they'll need to maintain at least 1900 customers weekly in order to remain in business. Will Megamart crush the new market and force it to close? If the weekly number of customers levels off eventually, how many customers will this be?

10) Among students who routinely buy dinner in either Harvard Square, Central Square, or Inman Square every Saturday night, we discover that of those who eat in Harvard Square, about 30% will go to Central Square next week and 10% will go to Inman Square. Of those who eat in Central Square, about 15% will go to Harvard Square next week and 15% will go to Inman Square. Of those who eat in Inman Square, about 25% will go to Harvard Square next week and 30% will go to Central Square.

After many weeks, what percentages of students will be eating in each of the three Squares? Find the exact proportions using eigenvalue-eigenvector analysis.

11) In 1940 a county land-use survey showed that 10% of the county land was urban, 50% was unused and 40% was agricultural. Five years later a follow-up survey revealed that 70% of the urban land had remained urban, 10% had become unused, and 20% had become agricultural. Likewise, 20% of the unused land had become urban, 60% had remained unused, and 20% had become agricultural. Finally, the 1945 survey showed that 20% of the agricultural land had become unused while 80% remained agricultural. Assuming that the trends indicated by the 1945 survey continue, compute the percentage of urban, unused, and agricultural land in the county in 1950 and the corresponding eventual percentages.

12) There are three interconnected rooms in a sleazy apartment. There are many flies buzzing about. During a long weekend, we find that after each hour 1/5 of the flies in the living room will have wandered into the kitchen and 1/5 into the bedroom. Of the flies that were in the bedroom, 1/3 have migrated into the kitchen and 1/6 into the living room. Of the flies in the kitchen the previous hour, 1/3 are now in the bedroom and 1/4 are in the living room. Assuming that the flies continue to exhibit this same behavior each hour (and that the population of flies doesn't change), how many total flies are there in the apartment if the number of flies in the living room is found to be holding steady at 130 flies? How many flies are in the kitchen and in the bedroom?

13) A country is divided into three regions. Each year, 10% of the residents of region 1 move to region 2 and 5% move to region 3; 15% of the residents of region 2 move to region 1 and 5% to region 3; and 10% of the residents of region 3 move to region 1 and 10% move to region 2. The total population of the country remains constant.

a) Find a matrix that represents the population transitions between regions from year to year.

b) Will the distribution of the population among the three regions tend toward fixed proportions? If so what shall these proportions be?

14) A video rental company has two locations, one in Boston and one in Cambridge, and borrowed videos can be returned to either location at no extra cost. There is an 80% probability that a video that is in Boston on Sunday will be in the Boston store the following Sunday, and a 70% probability that a video that is in Cambridge on Sunday will still be in Cambridge the following Sunday. Suppose the company stocks 10,000 videos which are initially all in Boston.

a) How will the videos be distributed between the two locations eventually? Set up the appropriate matrix equation and explain your reasoning.

b) How many videos do you expect at each location after 5 weeks?