HW 10 for Math 191. Due Tuesday Dec. 10.

1. Let X be a random variable. Show that its distribution function F_X is right continuous. That is, for any $c \in \mathbb{R}$ show that $\lim_{x \searrow c} F_X(x) = F_X(c)$.

2. Calculate the characteristic functions of:

   (a) The exponential distribution.

   (b) The uniform distribution on $[-a, a]$.

   (c) The binomial distribution.

   (d) The Poisson distribution.

3. Let X_n be random vars with binomial distribution $\{b(k; n, p_n)\}$ where $np_n \to \lambda \in (0, 1)$ as $n \to \infty$. Let Y have the Poisson distribution $\{p(k; \lambda)\}$. Use characteristic functions to show that the X_n converges to Y in distribution. Conclude that this gives another proof that for fixed k, one has $b(k; n, p_n) \to p(k; \lambda)$ as $n \to \infty$.

4. Let X be an integer valued random variable. Show that:

   (a) The characteristic function ϕ_X is periodic of period 2π.

   (b) For any $n \in \mathbb{Z}$ one has:

$$P\{X = n\} = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-int} \phi_X(t) dt.$$

5. Use characteristic functions to give a proof of the dreaded VII#9:

   Show that for any fixed α, β one has:

$$\sum_{\lambda+\alpha\sqrt{\lambda}<k<\lambda+\beta\sqrt{\lambda}} p(k; \lambda) \to \mathcal{N}(\beta) - \mathcal{N}(\alpha),$$

   as $\lambda \to \infty$.

6. Let X and Y be independent random vars with a common distribution that has mean 0 and variance 1. Prove that if $X + Y$ and $X - Y$ are independent then X (and thus Y) are normally distributed.

7. Let X and Y be independent random vars. Show that

$$\phi_{XY}(t) = \int_{-\infty}^{\infty} \phi_X(ty) dF_Y(y) = \int_{-\infty}^{\infty} \phi_Y(tx) dF_X(x).$$

8. Let X_n be the random var which corresponds to uniformly choosing a point in the interval $[-n, n]$.

   (a) Show that there is a function ϕ where $\phi_{X_n} \to \phi$ pointwise.

   (b) Prove that no subsequence of the X_n converges in distribution.

   (c) Explain why (a) and (b) do not contradict the continuity theorem.