Integration w.r.t. a prob. dist. "motivate."

Have prob dist on $\mathbb{R}$ (e.g. comming from a density fn)

Have $F(t) = P\{X \le t\} = P\{(-\infty, t]\}$ distribution function
increasing $0 \nearrow 1$

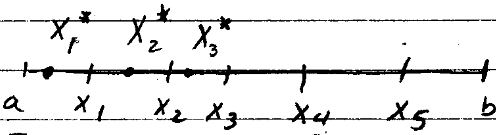
For $\alpha : [a,b] \longrightarrow \mathbb{R}$ will define

$$\int_a^b \alpha(x)\, dF(x)$$

" $= \int \alpha f\, dx$ for density fn."
" $= \sum \alpha(x_i) p_i$ for discrete."

Recall Riemann Sums: For $\int_a^b \alpha(x)\, dx$ we consider

$\mathcal{P} = \{a = x_0, x_1, x_2, \dots, x_n = b\}$



Let $\Delta x_i = x_i - x_{i-1} = \text{len}[x_{i-1}, x_i]$; Pick x_i^* in $[x_{i-1}, x_i]$

$$S(\alpha, \mathcal{P}, x_i^*) = \sum_{i=1}^{n} \alpha(x_i^*)\, \Delta x_i$$

So for $\int_a^b \alpha\, dF$: Let $\Delta_F x_i = P\{[x_{i-1}, x_i]\} = F(x_i) - F(x_{i-1})$

$$S_F(\alpha, \mathcal{P}, x_i^*) = \sum_{i=1}^{n} \alpha(x_i^*)\, \Delta_F x_i$$

Def: (Riemann-Stieltjes)

$$\int_a^b \alpha(x)\, dF(x) \text{ exists and equals } A$$

iff $\forall \varepsilon > 0$ $\exists$ a part $\mathcal{P}_\varepsilon$ s.t. $\forall \mathcal{P} \supseteq \mathcal{P}_\varepsilon$

we have
$$|S_F(\alpha, \mathcal{P}, x_i^*) - A| < \varepsilon$$

for every choice of x_i^*'s.

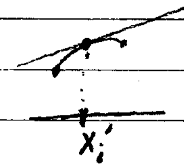
(For more, see Apostol, Math. Anal. Ch 7.)

Properties:

- Suppose F has a cont der. f (i.e. comes from a density fn).

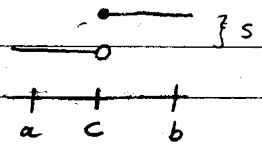
  Then $\int_a^b \alpha \, dF = \int_a^b \alpha f \, dx$.

  Idea: $\Delta_F x_i = F(x_i) - F(x_{i-1}) = f(x_i')\Delta x_i$



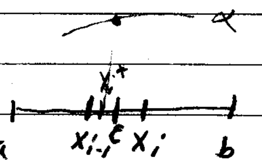
  Thus $S_F(\alpha, \mathcal{P}, x_i') = S(\alpha \cdot f, \mathcal{P}, x_i')$

- Step fns: $F =$



  Suppose α is cont at c, then $\int_a^b \alpha \, dF = s \cdot \alpha(c)$

  Pf:



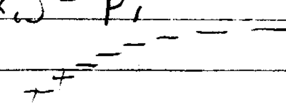
  $S_F = \alpha(x_i^*) \cdot \Delta_F x_i = \alpha(x_i^*) s$.

Uniform defn: Let $X : \Omega \to \mathbb{R}$ be a random var.

Then $E(X) = \int_{-\infty}^{\infty} x \, dF_X(x)$. (provided $\int_{-\infty}^{\infty} |x| \, dF_X(x) < \infty$)

density: $\int_{-\infty}^{\infty} x f_X(x) \, dx$

discrete: x_i $\quad P\{X = x_i\} = p_i$ $\quad F_X$



$\sum_{i=1}^{\infty} x_i p_i$

Convolution: X, Y are vars

$$F_{X+Y}(t) = F_X * F_Y \overset{def}{=} \int_{-\infty}^{\infty} F_X(t-x) \, dF_Y(x)$$

$$= \int_{-\infty}^{\infty} F_Y(t-x) \, dF_X(x).$$

Fun w/ gen fns:

$$X \text{ int valued}; \quad P(s) = \sum_{k=0}^{\infty} P\{X=k\} s^k = E(s^X)$$

- encodes dist of X, moments, etc
- independent → $P_{X+Y} = P_X P_Y \qquad E(s^{X+Y}) = E(s^X s^Y) = E(s^X s^Y)$

Moment gen fn: of X is $\psi_X(t) = E(e^{tX}) = E(s^X)\big|_{s=e^t}$

provided this expectation exists for all t in a nbhd of 0.

Problem: If ψ_X exists, then $E(|X|^k) < \infty \ \forall k.$

$$= \psi_X^{(k)}(0)$$

Solution: $\boxed{\text{Characteristic Functions}}$

Def: X rand var. The char fn is

$$\varphi_X(t) = E(e^{itX}) = \int_{-\infty}^{\infty} e^{itx}\, dF_X(x)$$

Notes: 1) Now we have complex valued vars. (so what?)

2) $\varphi_X(t)$ exist for any t:

$$\int_{-\infty}^{\infty} |e^{itX}|\, dF_X(x) = 1.$$

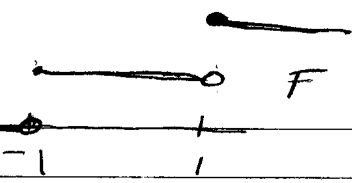
Ex: $X = c \Rightarrow \varphi_X(t) = e^{itX}$

Ex: X with normal dist

$$\varphi_X(t) = \int_{-\infty}^{\infty} \frac{1}{2\pi} e^{\overbrace{itx - x^2/2}^{= \frac{1}{2}(-(x-it)^2 - t^2)}} dx$$

$$= e^{-t^2/2} \underbrace{\int_{-\infty}^{\infty} e^{-(x-it)^2} dx}_{=1} = e^{-t^2/2}$$

Ex: $X = \{\pm 1\}$ w/ equal prob



$$\varphi_X = \int e^{itx}\, dF_X(x) = \tfrac{1}{2}\left(e^{-it} + e^{it}\right) = \cos t$$

notice is cont!

Properties: 1) $\varphi_X(-t) = \overline{\varphi_X(t)}$ 2) For indep X, Y, $\varphi_{X+Y} = \varphi_X \varphi_Y$.

3) φ_X is uniformly cont.

Pf:

$$|\varphi_X(t+h) - \varphi_X(t)| = |E(e^{i(t+h)X} - e^{itX})|$$

$$= |E(e^{it}(e^{ihX} - 1))| \le E(|e^{it}|)\, E(|e^{ihX} - 1|)$$

uniformity because doesn't dept on t $\rightarrow$ $= E(|e^{ihX} - 1|)$

As $h \to 0$, $\uparrow \to 0$ because:

$$|e^{ihX} - 1| \to 0 \text{ pointwise}$$

$\boxed{\text{and}}$

$$\underbrace{|e^{ihX} - 1|}_{\equiv g_h(x)} \le 2 \text{ for all } X.$$

So:

$$\int_{-\infty}^{\infty} g_h(x)\, dF_X = \underbrace{\int_{-N}^{N} g_h(x)\, dF_X}_{\text{for fixed } N \text{ this is small as } h \to 0} + \underbrace{\int_{|x|>N} g_h(x)\, dF_X}_{< 2(F_X(-N) + 1 -}$$

$\to 0$ as $N \to \infty$

Example of:

Lebesgue Dominated convergence thm.

Next time: φ_X determines F_X

- $X_n \to Y$ in dist $\Leftrightarrow$ $\varphi_{X_n} \to \varphi_Y$ pointwise.
- extracting moments from the φ_X.