Thursday, Dec 5.

From last time, question about

$$\int_{-T}^{T} \frac{e^{it(x-a)} - e^{it(x-b)}}{it}\, dt = 2\int_0^T \frac{\sin(t(x-a))}{t} - \frac{\sin t(x-b)}{t}\, dt$$

using $\dfrac{e^{it(x-a)}}{it} = \underbrace{\dfrac{\cos t(x-a)}{it}}_{\text{odd}} + \underbrace{\dfrac{\sin t(x-a)}{t}}_{\text{even}}$

Problem: The int of the cos term does not exist.

Solution: The combination of the two cos terms is actually continuous at 0 with value 0.

(Use L'Hopital rule to get val of integrand at 0)

Last time saw $\varphi_X = \varphi_Y \iff F_X = F_Y$ (i.e. X and Y have same dist.)

Continuity Thm: Let X_n, Y be rand vars.
Then $X_n \to Y$ in distribution iff $\varphi_{X_n} \to \varphi_Y$ pointwise.

Pf: ($\Rightarrow$) Fix t. Want $\varphi_{X_n}(t) \to \varphi_Y(t)$, i.e.

$$\int_{-\infty}^{\infty} e^{itx}\, dF_{X_n} \to \int_{-\infty}^{\infty} e^{itx}\, dF_Y$$

Given that $F_{X_n} \to F_Y$ at cont pts of F_Y and the above integrands are bounded, the above int limit is not hard to show.

($\Leftarrow$) Will give a sketch only. First note that it suffices to show that $F_Y(b) - F_Y(a) = \lim_{n\to\infty} F_{X_n}(b) - F_{X_n}(a)$ for any a, b which are cont pts of F_Y and all F_{X_n}.

$$F_Y(b) - F_Y(a) = \lim_{T\to\infty} \frac{1}{2\pi}\int_{-T}^{T} \underbrace{\frac{e^{-ita} - e^{-itb}}{it}}_{g(t)} \varphi_Y(t)\, dt$$

$$= \lim_{T\to\infty}\left(\frac{1}{2\pi}\int_{-T}^{T} g(t) \lim_{n\to\infty} \varphi_{X_n}(t)\, dt\right)$$

$$= \lim_{T\to\infty}\ \lim_{n\to\infty} \frac{1}{2\pi}\int_{-T}^{T} g(t)\,\varphi_{X_n}(t)\, dt$$

circle of omission

$$\text{“}=\text{”}\ \lim_{n\to\infty}\ \lim_{T\to\infty} \frac{1}{2\pi}\int_{-T}^{T} g(t)\,\varphi_{X_n}(t)\, dt$$

$$= \lim_{n\to\infty} F_{X_n}(b) - F_{X_n}(a). \qquad \blacksquare$$

(For a real proof see Billingsley Ch 26)

Application: $X_n = \{\pm 1\}$ w/ prob $1/2$.

$S_n = X_1 + \cdots + X_n$, $\quad S_n^* = \frac{S_n}{\sqrt{n}}$.

First occur of Central Limit Thm, back in rand walk days, was that $S_n^* \to$ normal dist. Let's do again, with charact fns:

$$\varphi_{X_n}(t) = \frac{e^{-it} + e^{+it}}{2} = \cos t.$$

$$\varphi_{S_n}(t) = \left(\varphi_{X_1}(t)\right)^n = \cos^n t$$

$$E(e^{it\alpha Y}) \quad E(e^{i(\alpha t)Y})$$

In general if $\alpha \in \mathbb{R}$, $\varphi_{\alpha Y}(t) = \varphi_Y(\alpha t)$ because.

$$\varphi_{S_n^*}(t) = \cos^n(t/\sqrt{n}) = \Big(1 - \frac{t^2}{2n} + \underbrace{\frac{t^4}{4!n^2} - \cdots}_{\text{neg.}}\Big)^n$$

$$\to e^{-t^2/2} \quad \text{as } n \to \infty.$$

So by the cont thm, $S_n^* \to$ normal distibution.

<u>Char Fns and Moments:</u>

<u>Thm:</u> If $E(|X|^k) < \infty$, then the derivative $\varphi_X^{(k)}$ exists and $E(X^k) = i^{-k}\varphi_X^{(k)}(0)$

<u>Pf:</u> Formally

$$\varphi_X^{(k)}(0) = \frac{d^k}{dt^k}\varphi_X\Big|_{t=0} = \frac{d^k}{dt^k}\Big[\int_{-\infty}^{\infty} e^{itx}\,dF_X\Big]\Big|_{t=0}$$

$$\text{"="} \int_{-\infty}^{\infty} \frac{d^k}{dt^k} e^{itx}\Big|_{t=0}\,dF_X = \int_{-\infty}^{\infty} (ix)^k\,dF_X$$

$$= i^k E(X^k)$$

The only issue is whether we can diff under the inegral sign. Certainly, for such a thing to be poss, need that this integral exists; turns out that that is enough. Thus our hyp. suffice to prove the theorem. ∎

Cor: Suppose that $E(|X|) < \infty$. Then

$$\varphi_X(t) = 1 + iE(X)t + o(|t|)$$

(here f is $o(|t|)$ if $f(t)/|t| \to 0$ as $t \to 0$.)

Pf: By thm, $\varphi_X'(0)$ exists and $= iE(X)$. ∎

Cor: Suppose that $E(|X|^2) < \infty$. Then:

$$\varphi_X(t) = 1 + iE(X)t - \frac{E(X^2)}{2}t^2 + o(|t|^2)$$

Central Limit Thm: Let X_i be indep vars w/ a common dist w/ mean μ and var σ^2. Then if $S_n = X_1 + \cdots + X_n$ and $S_n^* = \frac{S_n - n\mu}{\sigma\sqrt{n}}$, then $S_n^* \to$ var with normal distribution.

Pf: WLOG, assume X_i have $\mu = 0$. Then

$$\varphi_{S_n^*}(t) = \varphi_{S_n}\left(t/\sigma\sqrt{n}\right) = \varphi_{X_1}\left(t/\sigma\sqrt{n}\right)^n$$

$$= \left(1 - \frac{\sigma^2}{2}\left(\frac{t}{\sigma\sqrt{n}}\right)^2 + o\left(\frac{t^2}{\sigma^2 n}\right)\right)^n$$

$$= \left(1 - \frac{t^2}{2n} + o\left(\frac{1}{n}\right)\right)^n \longrightarrow e^{-t^2/2}.$$

So by the continuity thm, $S_n^* \to$ normally dist var. ∎

Thm Suppose Z has mean 0 and var 1.
Suppose that if Z_1 and Z_2 are indep, dist by Z,
then $\dfrac{Z_1 + Z_2}{\sqrt{2}}$ has the same dist as Z.
Then Z is normally dist

Pf: Note

$$\varphi_{\frac{Z_1+Z_2}{\sqrt{2}}}(t) = \varphi_{Z_1+Z_2}(t/\sqrt{2}) = \varphi_Z(t/\sqrt{2})^2$$

($\Leftarrow$) If Z is normal, $\varphi_Z = e^{-t^2/2}$ so

$$\varphi_{\frac{Z_1+Z_2}{\sqrt{2}}}(t) = \varphi_Z(t/\sqrt{2})^2 = e^{-t^2/2} = \varphi_Z(t)$$

By uniq. theorem, $(Z_1+Z_2)/\sqrt{2} \overset{dist}{=} Z$.

($\Rightarrow$) Then $\varphi_Z(t/\sqrt{2})^2 = \varphi_Z(t)$. So inductively,

$$\varphi_Z(t) = \varphi_Z(t/2^{n/2})^{2^n}$$

$$= \left(1 - \frac{t^2}{2 \cdot 2^{n/2}} + o(|t|^2)\right)^{2^n}$$

$$\to e^{-t^2/2} \quad \text{as } n \to \infty.$$

So $\varphi_Z(t) = e^{-t^2/2}$ and so $Z = N(0,1)$