Tuesday Dec 3:

Last time, introduced Characteristic Functions

X rand var, F_X corresp. dist fn

Def (Char fn)

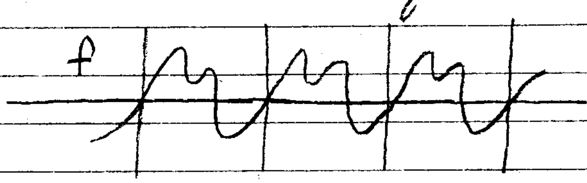
$$\varphi_X(t) = E(e^{itX}) = \int_{-\infty}^{\infty} e^{itx}\, dF_X(x)$$

Prop (from last time): · Sums of indep vars $\Rightarrow$ prod of char fns.
· φ_X uniformly cont, regardless of F_X.

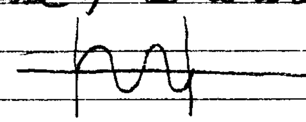
In non-probabilistic contexts, φ_X is called the Fourier transform.

Some motivation: Fourier trans. as freq decomp of wave.

Consider a cont function f, with $f(x) = f(x+2\pi)$ for $\forall x$.



Want to decompose f as a sum of simple waves



w/ period dividing 2π.

"Could use sines/cosines, but simpler to introduce complex #s and"

$$(*)\quad f(x) = \sum_{n=-\infty}^{\infty} a_n e^{inx} \qquad a_n = \text{"prop of } f \text{ at freq } 2\pi/n\text{."}$$

where can find a_n by integration

$$\int_0^{2\pi} e^{ikx} f(x)\, dx = \sum_{n=-\infty}^{\infty} \Big[a_n \underbrace{\int_0^{2\pi} e^{i(k+n)x}\, dx}_{=0 \text{ unless } n=-k} \Big]$$

$$= a_{-k}$$

To compare w/ char fn, note that when F_X has a density fn,

$$\varphi_X(t) = \int_{-\infty}^{\infty} e^{itx} f(x)\, dx \quad \text{which is similar to}$$

Without the period. assump on f, we int in all values of t, not just integral ones.

Also, there is an analogue to (*) but now we need an integral, not a sum:

Fourier Inversion Thm: If $\int_{-\infty}^{\infty} |\varphi_X(t)|\,dt < \infty$, then F_X has a cont density given by:

$$f_X(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-itx}\varphi_X(t)\,dt.$$

"Note: I_T is not written as $\int_{-\infty}^{\infty}$ because that int may not exist."

Thm: X rand var. For any a, b where F_X is cont

$$P\{a < X < b\} = \lim_{T\to\infty} \underbrace{\frac{1}{2\pi}\int_{-T}^{T} \frac{e^{-ita} - e^{-itb}}{it}\varphi_X(t)\,dt}_{I_T}$$

Cor: X, Y vars. If $\varphi_X = \varphi_Y$ then $F_X = F_Y$.

("cor follows from thm as F_X, F_Y have only count. many singularities and are right cont" ~ on HW.)

Pf:

$$I_T = \int_{-T}^{T}\int_{-\infty}^{\infty} \frac{e^{it(x-a)} - e^{it(x-b)}}{it}\,dF_X(x)\,dt$$

"ok to switch order as int over set of finite measure + bounded integrand"

$$= \int_{-\infty}^{\infty}\left[\int_{-T}^{T} \frac{e^{it(x-a)} - e^{it(x-b)}}{it}\,dt\right] dF_X(x)$$

When we expand out, we'll get an ans in terms of

$$S(T) = \int_0^T \frac{\sin x}{x}\,dx \qquad T \geq 0.$$

Which has

$$\lim_{T\to\infty} S(T) = \pi/2$$

(This is left as exercise, but if all we care about is the Cor it doesn't matter what this lim is)

Also, for any $\alpha \in \mathbb{R}$ if $sgn(\alpha) = \begin{cases} 1 & \alpha > 0 \\ 0 & \alpha = 0 \\ -1 & \alpha < 0 \end{cases}$ ③

$$\int_0^T \frac{\sin \alpha x}{x}\, dx = \int_0^T sgn(\alpha)\, \frac{\sin |\alpha| x}{x}\, dx$$

$$= \int_0^T sgn(\alpha)\, \frac{\sin|\alpha|x}{|\alpha| x}\, |\alpha| dx = sgn(\alpha)\, S(|\alpha| T).$$

Returning to I_T, note

$$\frac{e^{-it(x-a)}}{it} = \underbrace{\frac{\cos(t(x-a))}{it}}_{\text{odd}} + \underbrace{\frac{\sin(t(x-a))}{t}}_{\text{even}}$$

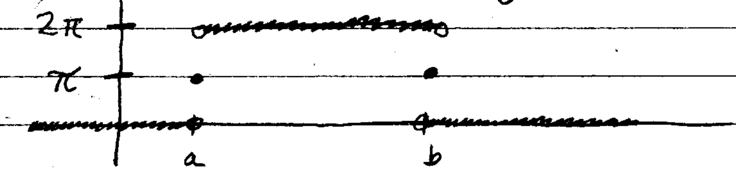
So

$$\int_{-T}^{T} \frac{e^{-it(x-a)} - e^{it(x-b)}}{it}\, dt = 2\int_0^T \frac{\sin(t(x-a))}{t} - \frac{\sin(t(x-b))}{t}\, dt$$

$$= 2\big(sgn(x-a)\, S(|x-a| T) - sgn(x-b)\, S(|x-b| T)\big)$$

The above ↑ is bounded and converges as $T \to \infty$ to

$\psi_{a,b}$:



So

$$I_T \longrightarrow \int_{-\infty}^{\infty} \psi_{a,b}\, dF_X \quad \text{as} \quad T \longrightarrow \infty.$$

By assumption, F is cont at a, b (equiv $P\{X=a\} = P\{x=b\} = 0$)

so $$\int_{-\infty}^{\infty} \psi_{a,b}\, dF_X = \int_a^b 2\pi\, dF_X = 2\pi \cdot P\{a < X < b\}$$

This proves the theorem. ∎

Now for useful special case mentioned earlier.

Thm: Suppose $\int_{-\infty}^{\infty} |\varphi_X| < \infty$. Then F_X has a cont density function given by:

$$f_X(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-itx}\varphi_X(t)\,dt$$

Pf: Since $\int |\varphi_X| < \infty$, the can get rid of the limit:

$$P\{a < X < b\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{e^{-ita} - e^{-itb}}{it}\varphi_X(t)\,dt$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\int_a^b e^{-itx}\,dx\right]\varphi_X(t)\,dt$$

$$= \int_a^b \left[\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-itx}\varphi_X(t)\,dt\right]dx. \quad \blacksquare$$

Remark: Can use this ver to see $\lim_{T\to\infty} S(T) \equiv S(\infty) = \pi/2$.

Note what the Thm showed was $P\{a<X<b\} = \frac{1}{4S(\infty)}\lim_{T\to\infty} I_T$.

Know for normally dist, $\varphi_X(t) = e^{-t^2/2}$

By above:

$$f_X = \frac{1}{\sqrt{2\pi}}e^{-x^2/2} = \frac{1}{4S(\infty)}\int_{-\infty}^{\infty} e^{-itx - t^2/2}\,dt$$

$$= \frac{\sqrt{2\pi}}{4S(\infty)}\cdot\left[\frac{1}{\sqrt{2\pi}}\int e^{-itx-t^2/2}\,dt\right] = \frac{\sqrt{2\pi}}{4S(\infty)}$$

$\Rightarrow S(\infty) = \pi/2.$

Reference for this lecture: Billingsley; Prob and Measure. Section 26.