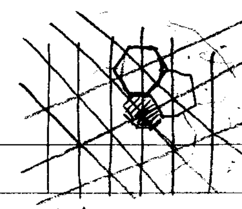
Thursday) Dec 17:

Percolation: $\mathcal{L}$ a lattice, for today  equitriangular

Usually draw "fat vertices" hexagon picture

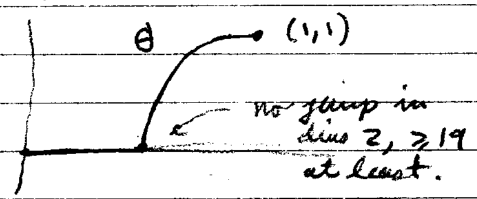
Color verts black w/ prob p white w/ prob $1-p$.
Interested in connected clusters.

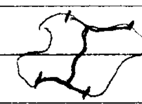
Set $\Theta(p) =$ prob that a fixed vertex is part of an infinite black cluster. || increasing function in p, $\theta(0)=0$, $\theta(1)=1$

Analogy of immersing porous rock in water.

Thm: For a fixed lattice $\mathcal{L}$, $\exists p_c \in (0,1)$ s.t.

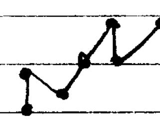
$$\Theta(p) = \begin{cases} 0 & p < p_c \\ 1 & p > p_c \end{cases}$$



(Same p_c as talked about last time )

Pf: A: $\exists\, p > 0$ w/ $\Theta(p) = 0$.

Fix base vertex at origin. Let

$\sigma(n) =$ number of self avoiding paths in $\mathcal{L}$ of len n starting at 0 

Note $\sigma(n) \leq 6 \cdot 5^{n-1}$ and $\sigma(n) \geq 6 \cdot 2^n$

$$\lambda = \lim_{n\to\infty} \sigma(n)^{1/n} \qquad \lambda > 0$$

is the exponential growth rate of $\sigma(n)$.

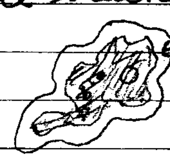
Now:

$$P\{0 \text{ is part of an inf cluster}\} \leq P\left\{\begin{array}{l}\exists \text{ some all black} \\ \text{path of len } n \\ \text{starting at the} \\ \text{origin}\end{array}\right\}$$

$$\leq p^n \sigma^n \leq p^n \cdot 6 \cdot 5^{n-1}$$

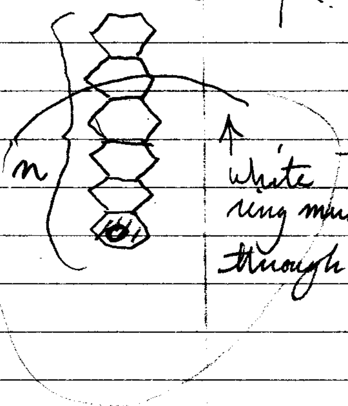
If $p < \frac{1}{5}$, then $p^n \sigma^n \to 0$ as $n \to 0$.

So for $p < 1/5$, $\Theta(p) = 0$ as required.

B: $\exists\, p < 1$ w/ $\Theta(p) > 0$. Suppose 0 is not part of an infinite black cluster. Then $\exists$ a white ring around it: 

Set $\rho(n) = \#$ of paths surrounding 0 of length n.

$$\leq n \cdot \sigma(n-1) \leq n\, 6 \cdot 5^{n-1}$$



So:

$$P\{0 \text{ not in inf cluster}\}$$

$$\leq \sum_{n=0}^{\infty} (1-p)^n \rho(n) \leq \sum_{n=0}^{\infty} (1-p)^n n \cdot 6 \cdot 5^{n-1}$$

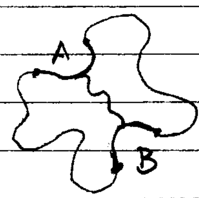
for $1-p$ small enought is $\leq 1/2$. Thus $\Theta(\text{such } p) > 0$.

Now that we have A and B, thm follows by taking

$p_c = \sup\{p \mid \Theta(p) = 0\}$, by monotonicity. ∎

Thm (Kesten 1980) $p_c = 1/2$.

Thm: Let $U \subseteq \mathbb{C}$ be a region bounded by a closed curve with marked intervals A, B. Set

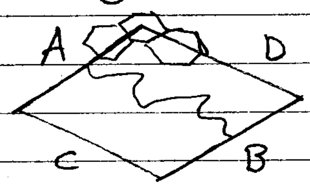


$$\pi(p) = \lim_{\text{mesh size} \to 0} P\left\{\begin{matrix}\exists \text{ a black path from} \\ \text{joining } A \text{ to } B\end{matrix}\right\}$$

Then
$$\pi(p) = \begin{cases} 0 & p < p_c \\ 1 & p > p_c \end{cases}$$

Of 2^{nd} thm ($p > p_c \Rightarrow$ crossing prob 1) is easy direction.

Assuming 2^{nd} thm, can calculate p_c using symmetry



Note $\exists$ a black path joining A to $B \Leftrightarrow \nexists$ a white path joining C to D.

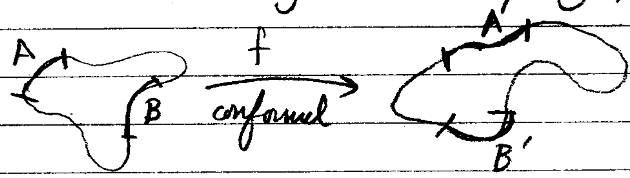
Thus $P\{\text{black } A \text{ to } B\} + P\{\text{white } C \text{ to } D\} = 1$

At $p = 1/2$, these are same, so $P\{\text{black } A \text{ to } B\} = 1/2$.

$\Rightarrow p_c = 1/2$. (Comment on John Nash, hex, etc.)

<u>Critical Percolation</u>: Crossing probs at p_c.

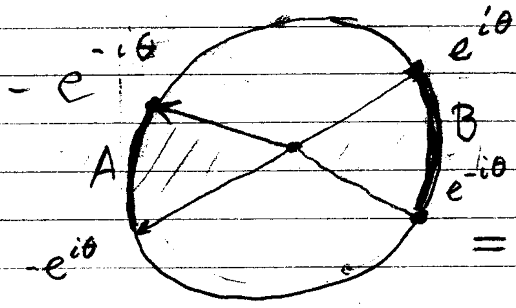
<u>M. Aizenmann</u>: Conj that crossing prob is conformally invariant.



In particular in figure

$P\{\text{crossing } A \to B\} = P\{\text{crossing } A' \text{ to } B'\}$

<u>J. Cardy 1992</u>: Conf. invariance lets you compute the crossing probs explicitly.



Crossing prob of $(1, r)$ rectangle is

$$= \frac{\Gamma(\frac{2}{3})}{\Gamma(\frac{4}{3})\Gamma(\frac{1}{3})} \eta^{1/3} \; {}_2F_1(1/3, 2/3, 4/3; \eta)$$

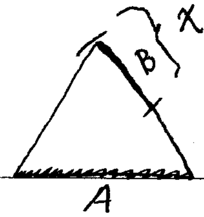
where

$$\eta = \sin^2\theta = \frac{(z_1 - z_2)(z_3 - z_4)}{(z_1 - z_3)(z_2 - z_4)}$$

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} \, dt \qquad \Gamma(n) = (n-1)!$$

${}_2F_1$ = hypergeometric function

Carleson's version:

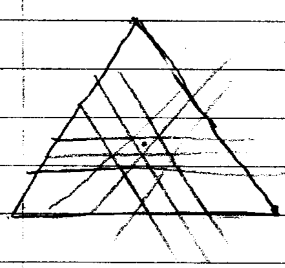


unit side equilateral triangle

$$P\{\text{crossing } A \to B\} = x.$$

1994: Langlands, Pouliot, Saint-aubin: verified Cardy's formula approx in computer experiments.

2001: S. Smirnov proves conformal invariance for $\mathcal{L}$ = equitriangular lattice.



Idea: Consider w/ mesh of size δ.

Define $H^{\delta}(p) =$

Prob($\exists$ a black path)

For $p \in$ this is the crossing formula for Carleson's model

Shows that as $\delta \to 0$, $H^{\delta} \to H$. Smirnov shows:

a) H is harmonic $\left(\Delta H = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) H = 0 \right)$

b) $H(\triangle) = 1 \qquad H(\triangle) = 0$

c) on $\triangle$ the comp of ∇H which is $\parallel$ to $\triangle$ vanishes.

$\Longrightarrow$ H is linear for sat c).

Do in general using invariance of harmonic fns under conformal maps.

The End.