Thursday, Dec 12

Conformal Mappings: $U^{\text{open}} \subseteq \mathbb{C}$. Then $f: U \longrightarrow \mathbb{C}$ is conformal if the derivative at every pt preserves angles.
(+ orient pres, + der non-zero).

Ex: Euclidean motions; dilations; $z \longmapsto 1/z$ (inversion in ○)
In general, holomorphic maps.

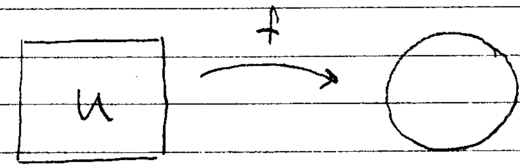
Non Ex: $(x,y) \longmapsto (2x, y)$.

Another characterization: Conformal = infinitesimal circles go to infinitesimal circles

Riemann Mapping Theorem: Let U be a proper open subset of $\mathbb{C}$, which is simply connected.
Then $\exists$ a conformal map $f: U \longrightarrow D = \{ z \in \mathbb{C} \mid |z| < 1 \}$

which is 1-1 and onto.

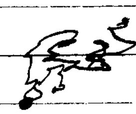
Note: The pt is not topological.



· Show slides, discuss

Brownian Motion in 2D: U_t, V_t indep Brownian motions

$$W_t = U_t + iV_t$$



Last time: Rotational invariance.

In 1-d case saw if U_t is Brownian motion, so is

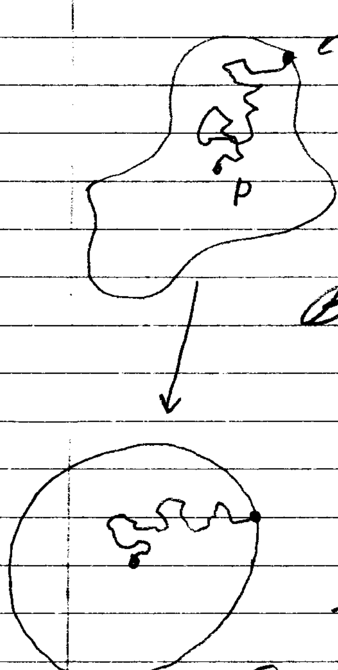
$$U'_t = c\, U_{c^{-2}t}$$; i.e, is inv under dialation after rescaling time.

Now in 2-d consider

$$W'_t = c\, W_t$$

By above, this is just 2-d Brownian motion, except that we've changed how fast things move.

Let $U \subseteq \mathbb{C}$ be open, $p \in U$. Consider constrained Brownian motion W_t in U starting at p.



Stops when hits the boundary.

Let $f : U \to V$ be conformal.

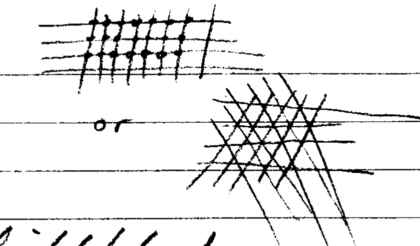
Then $f(W_t)$ is a time rescaled Brownian motion.

<u>Why care?</u> Additions of symmetries in the limit lead to greater understanding. (refer back to slide)

<u>Ex:</u> Hitting probabilities.

<u>Aside:</u> Conformal invariance of Brown Motion gives another proof of the fund thm of Algebra.

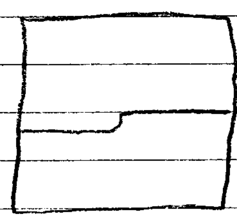
Percolation: Start w/ a lattice L, e.g.



Color vertices black and white w/ prob p and $1-p$. Interested in regions of light/dark.

- Model for liquids in porous media.
- Undergo phase transition at a critical probability.

Simple question: Crossing probabilities.



Does there exist a black path from left to right?
Let $\pi_n^n(p)$ be this prob for an $n \times n$ square lattice w/ black prob p.

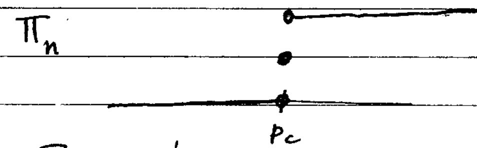
Rmk: For fixed n, $\pi_n^n(p)$ is an increasing fn of p.
Rmk: Elec. conduct of a random media.
Show Graph of $\pi_n^n(p)$

If we take the limit as $n \to \infty$, there is a critical probability p_c at which the crossing prob jumps from 0 to 1.

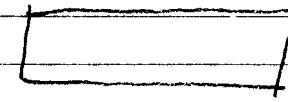
For square lattice

$$p_c \approx 0.59 \quad \text{(no exact formula is known!)}$$

Thm (Kesten 1980) For any lattice, there is a critical prob p_c w/ the transition prob seen above.



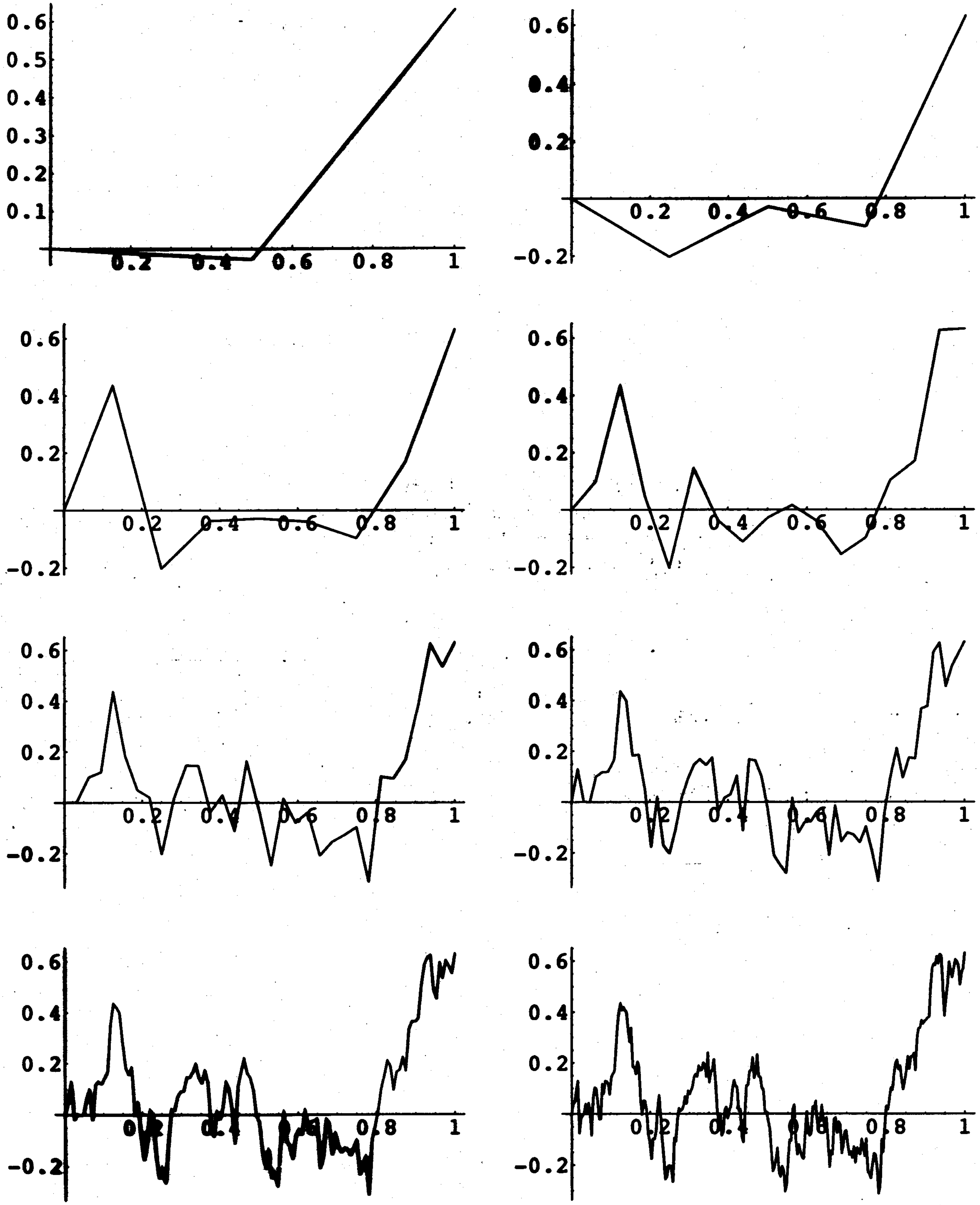
Q: At p_c what is π_n? How about for a rectangle? A general region?



Next time: Conformal invariance of percolation (Smirnov 2001)

Cardy's crossing formula.

$p_c = 1/2$ for triangular lattice (and John Nash).



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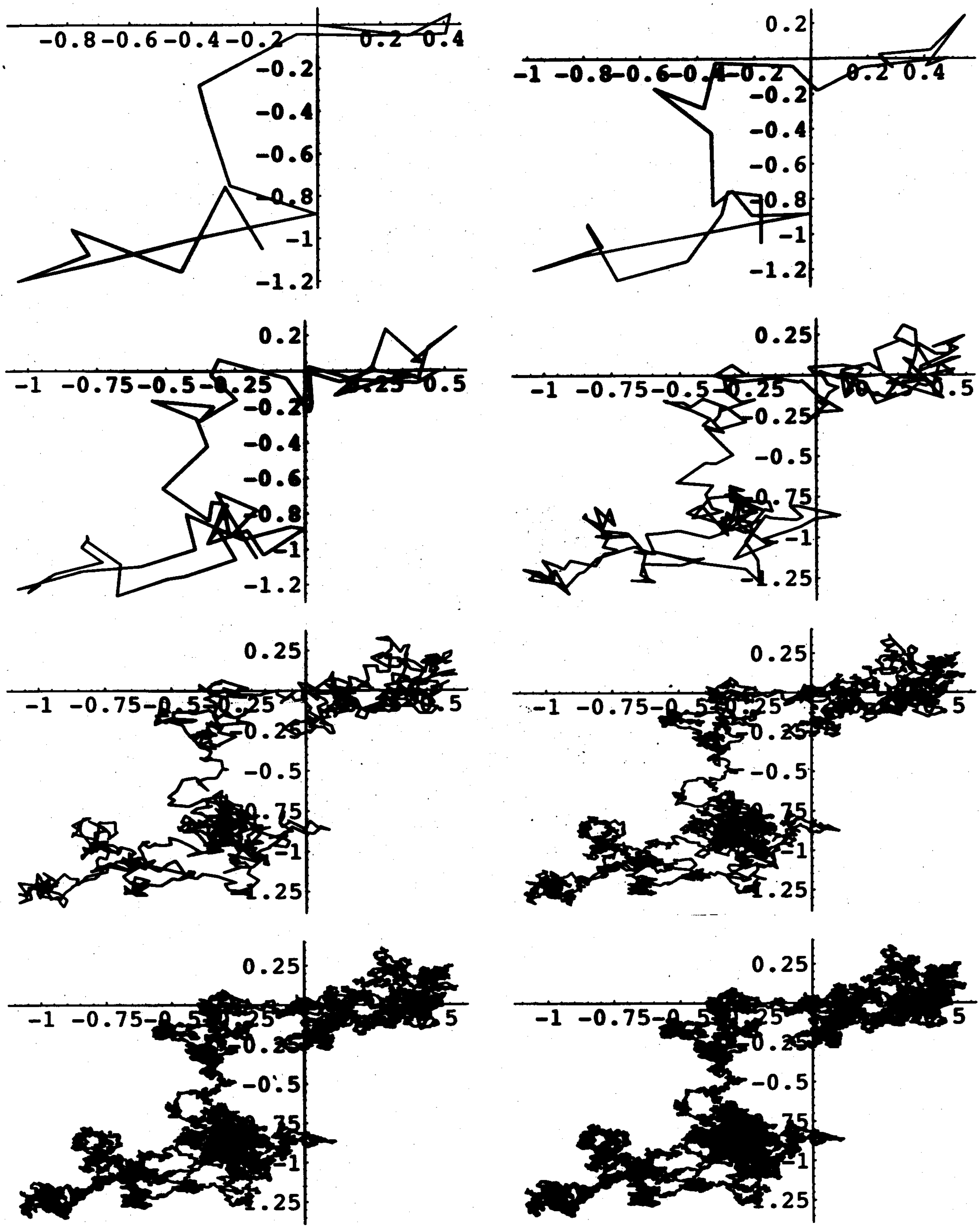

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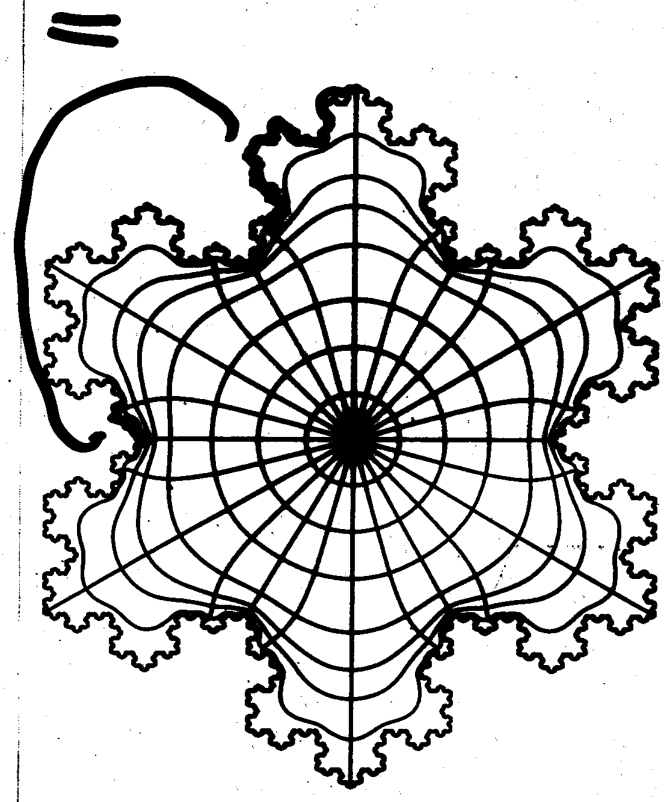
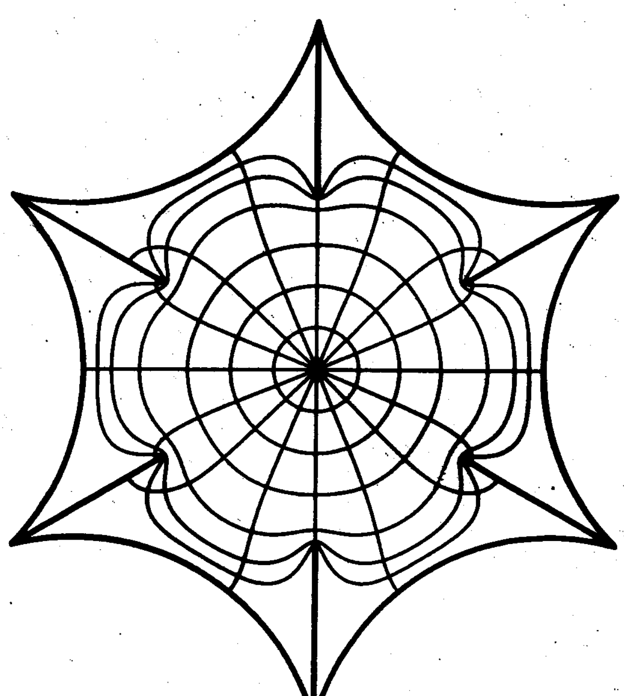
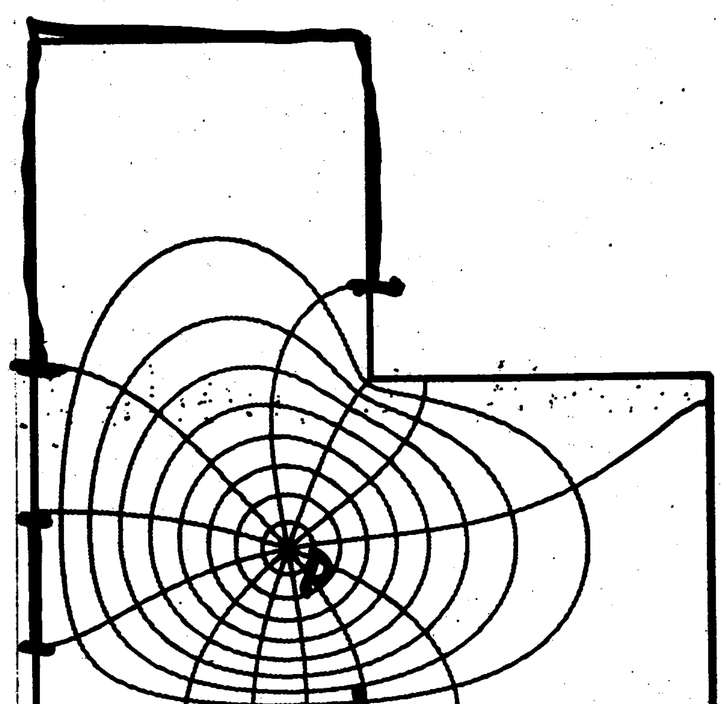






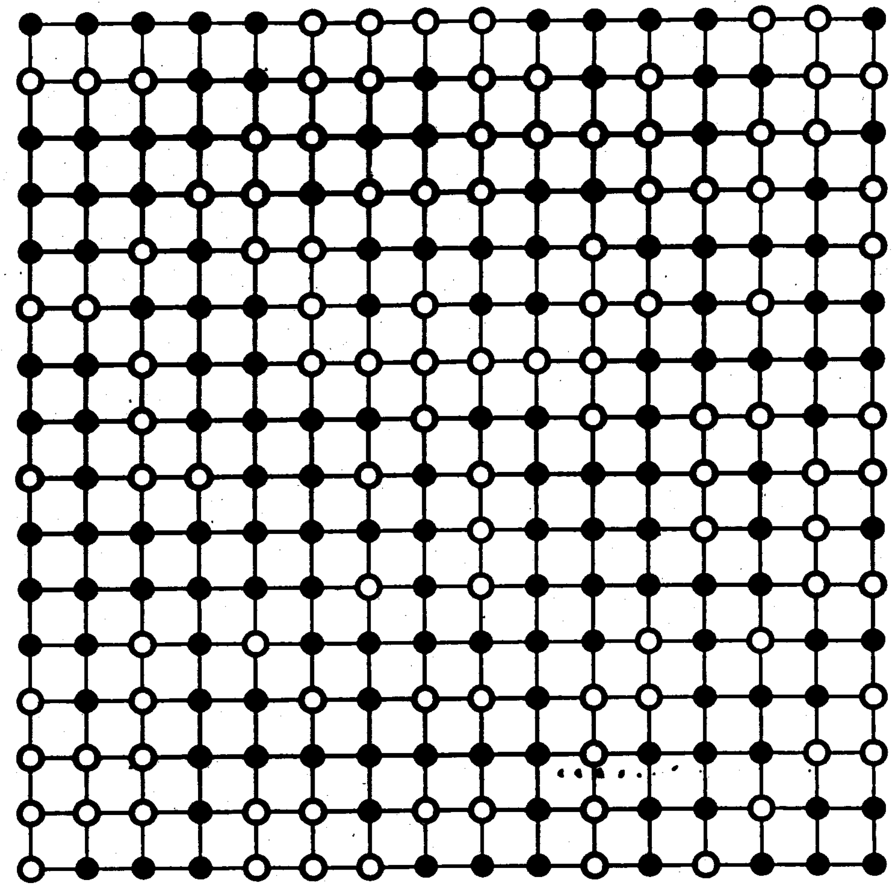
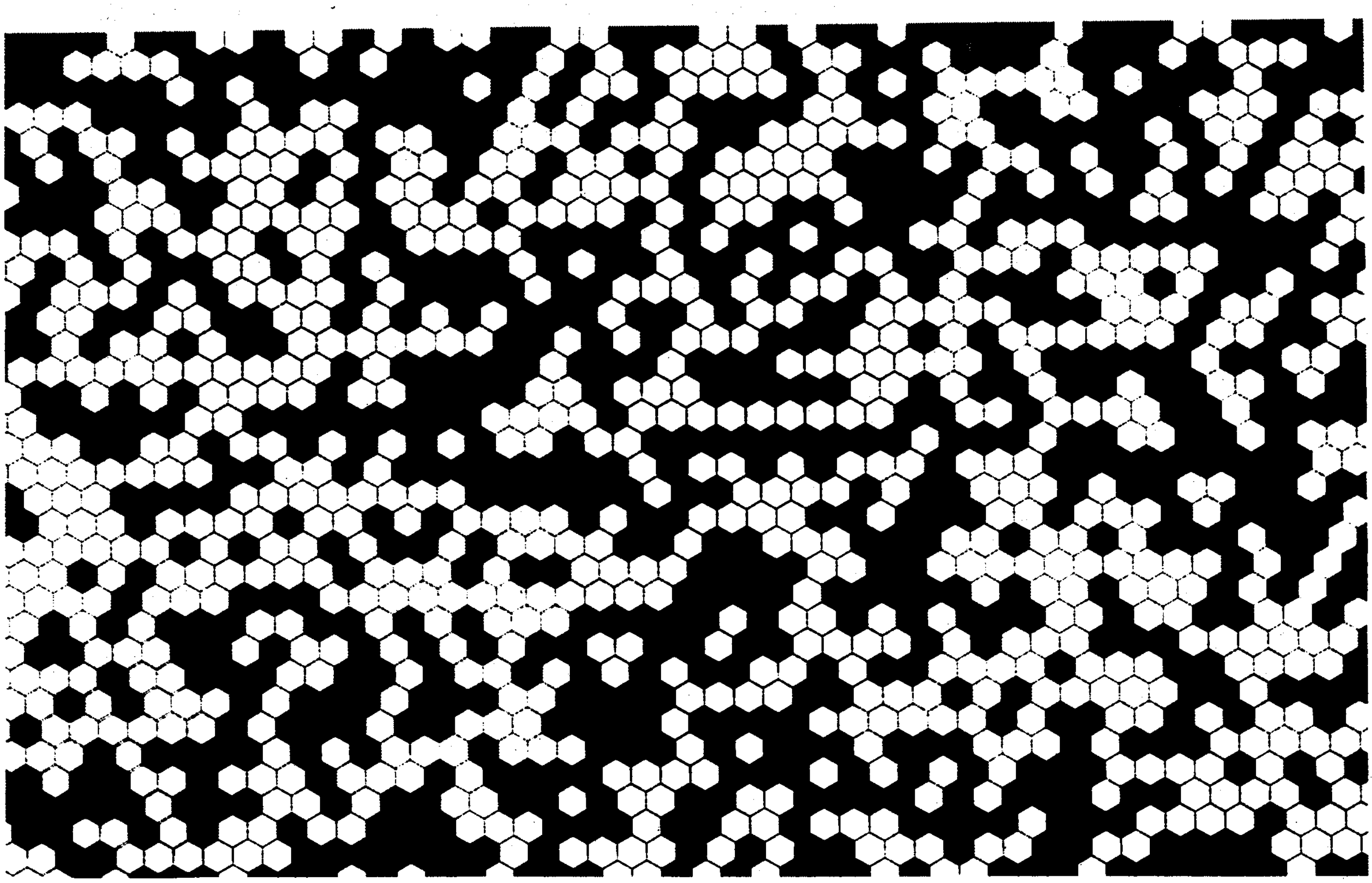


FIGURE 2.1a. Configurations on the square cube S_{16} for percolation by sites.



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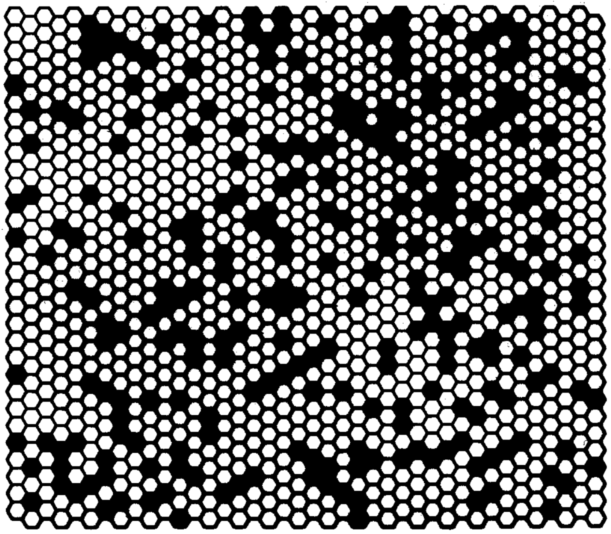

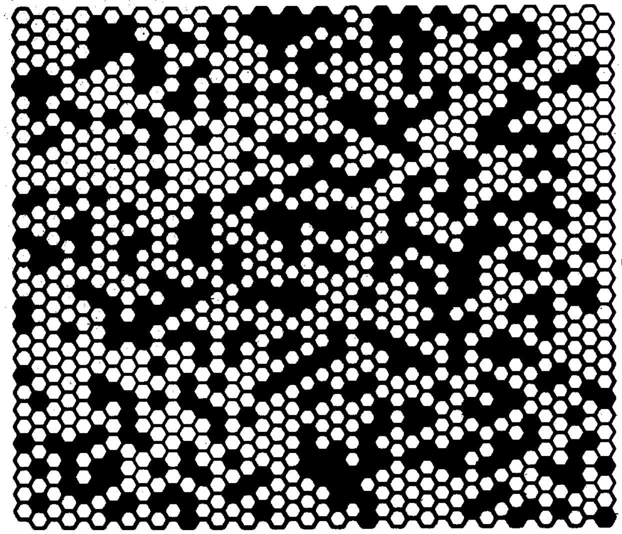

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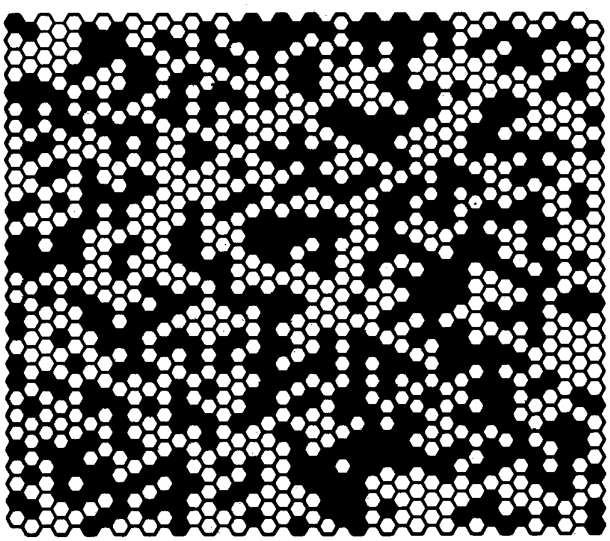

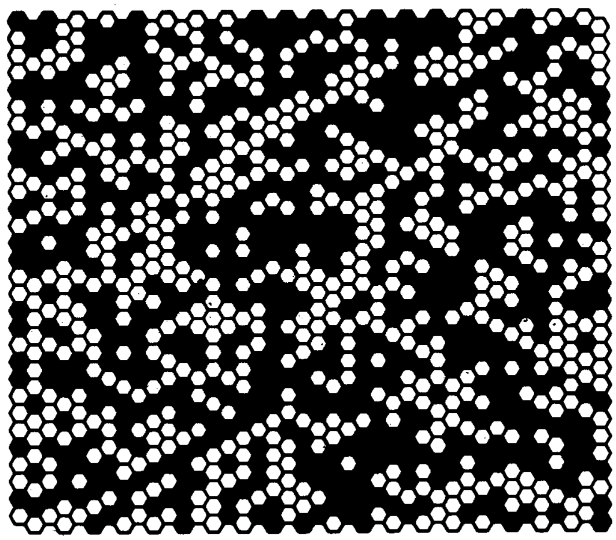

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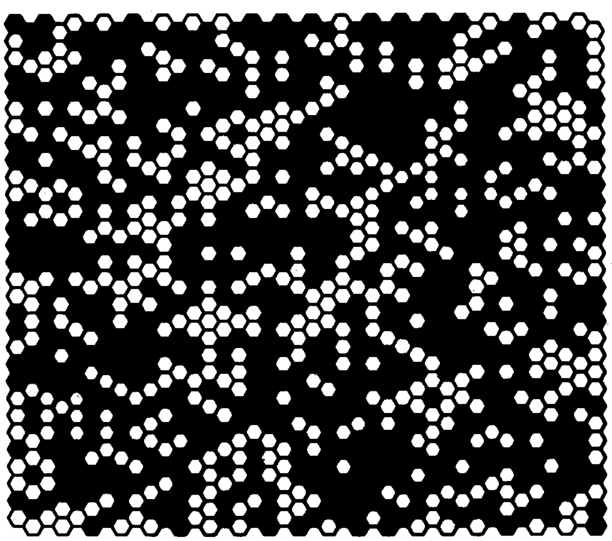

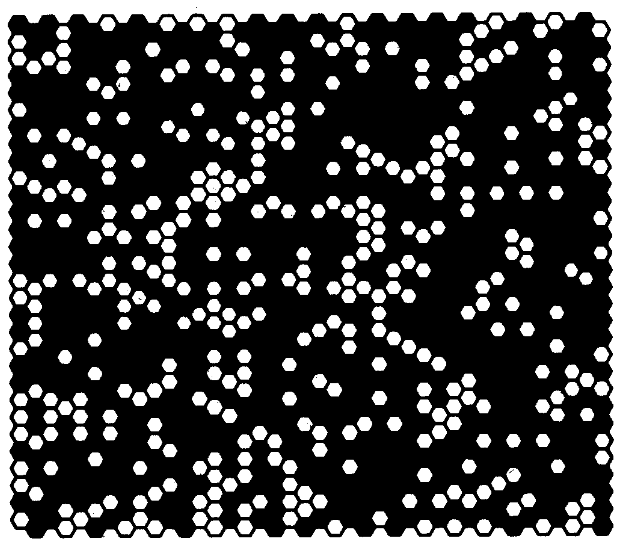

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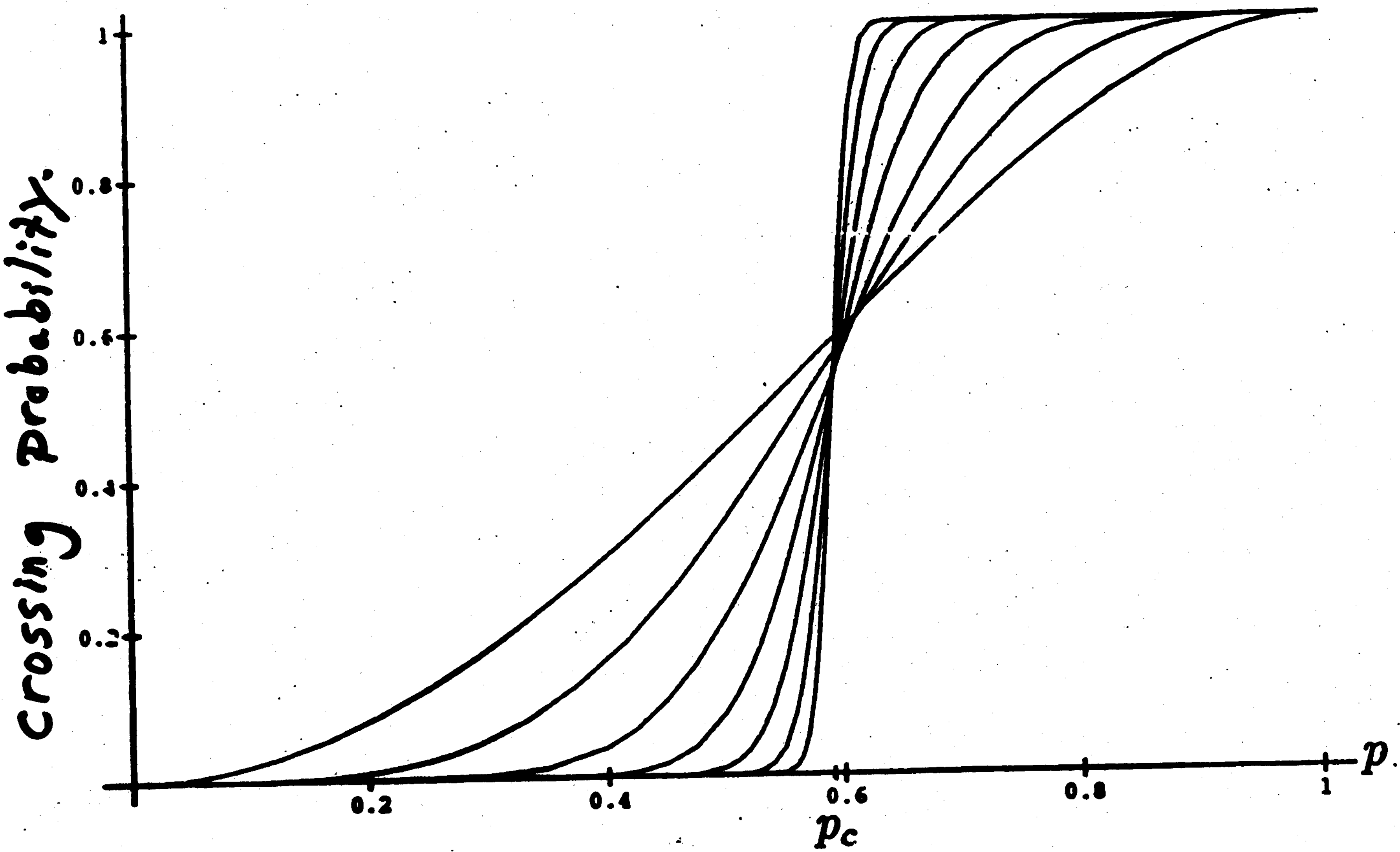

FIGURE 2.1c. The curves $\pi_h^n(p)$ for $n = 2, 4, 8, 16, 32, 64$, and 128. Larger slopes around p_c correspond to larger values of n.