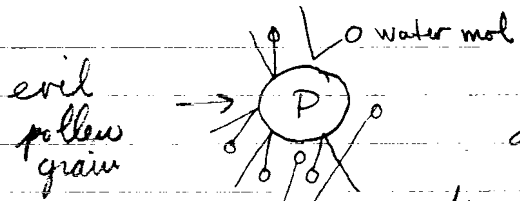
Tuesday Dec 10:

Brownian Motion.

Robert Brown in 1827: pollen grains suspended in liquid. Erratic motion.



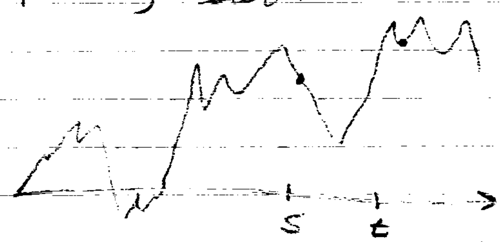
Thermal motion of liquid results in collisions between water moll. and pollen grain.

Simplify to 1-d. Because of Central Limit Theorem, motion should follow normal dist. Moreover, the var grows linearly w/ time, because of lack of memory.

Def: A Brownian motion is a sample space Ω (w/prob) with random vars $\{W_t \mid t \geq 0\}$ sat:

i) The path starts at 0:

$$P\{W_0 = 0\} = 1$$



ii) Markov Prop, increments are indep: If

$0 \leq t_0 < t_1 < \cdots < t_k$ and $H_0, \dots, H_k \subseteq \mathbb{R}$ then

$$P\{W_{t_i} - W_{t_{i-1}} \in H_i \text{ for all } i\} = \prod_{i=1}^{k} P\{W_{t_i} - W_{t_{i-1}} \in H_i\}$$

iii) For $0 \leq s < t$ the increment $W_t - W_s$ is normally dist with mean 0 and var $t - s$.

Note: Density fn of $N(\mu, \sigma)$ is

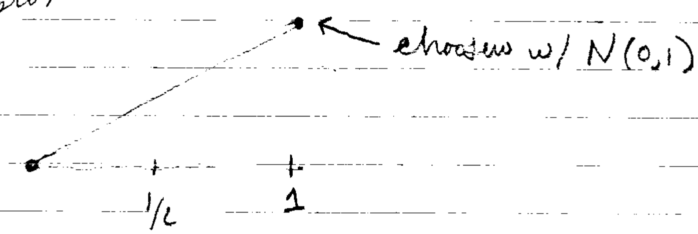
$$\frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/2\sigma^2}$$

Thm: Brownian Motion exists.

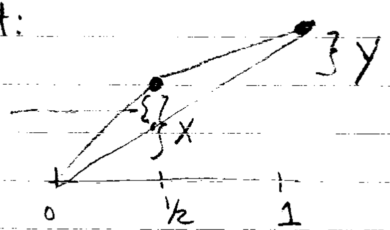
Thm: With prob 1, a path $W_t(s)$ of Brownian motion is continuous.

Constructing Brownian Motion: on $[0,1]$

First approx:



Choosing next pt:

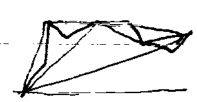


X, Y indep, dist like $N(0, 1/\sqrt{2})$

Given $X+Y$

Set $Z = X - \frac{X+Y}{2} = \frac{1}{2}(Y-X)$. By HW, Z is indep of $X+Y$ since X, Y are normal dist. Moreover, Z is dist like $N(0, 1/2)$. So set $W_{1/2} = W_1 + N(0, 1/2)$

Continue refining



When refining interval of length 2^{-n}, perturb midpoint $N(0, 2^{-n/2-1})$.

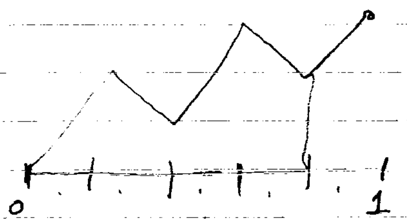
This defines W_t on the dyadic rationals $= \{\frac{k}{2^n} \mid k, n \in \mathbb{Z}\} = \mathcal{D}$

Can show that, w/ prob 1, the path is uniformly cont on $\mathcal{D}$ $\Rightarrow$ has a unique extension to a cont fn.

Usually, use model where the sample space for the W_t is $C([0,1])$ or $C([0,\infty))$ (the space of cont fns on the corr. interval)

<u>Scaling Limit:</u> Consider a rand walk w/ 2^n steps in $[0,1]$ w/ step size $2^{-n/2}$ (so $\sum \text{var} = 1$).

$$S_n = X_1 + \cdots + X_{2^n} \qquad X_k = \{\pm 2^{-n/2}\}$$



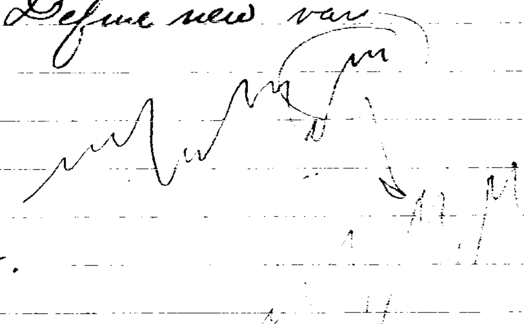
For fixed n, let 2^n cont paths i.e. pts in $C([0,1])$. If take the assoc prob P_n on $C([0,1])$ these conv. weakly to Brownian motion.

<u>Properties:</u>

1) Many sim to random walk (still have reflection principle)

2) <u>Statistical-self similarity:</u> Fix c. Define new var

$$V_t = c^{-1} W_{c^2 t}$$



Then V_t is again Brownian motion.

3) <u>Irregularities:</u> With prob 1, a Brownian motion path is differentiable nowhere.

④ Why? $\exists c$ s.t. $W|_{[0,c]}$ has a chord of slope ≥ 1 w/ high prob. Then $V|_{[0,c^{-1}]}$ has a chord w/ slope c w/ high prob.. So must great slopes on small scales.

Thm: With prob 1, the set of local max is dense.

Thm With prob 1, the set $Z = \{t \mid W_t = 0\}$ is closed, unbounded, has no isolated pt in $(0,\infty)$. (Cantor set)

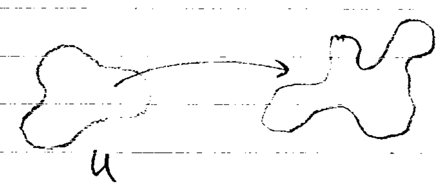
Brownian Motion in 2^d: Let U_t, V_t be two indep Brownian motions. Set

$$X_t = U_t + i V_t$$

Notes: Dist $X_t - X_s$ is a sym. Gaussian one. so doesn't dep on choice of orthonormal axis

Conformal Invariance of Brown Motion:

Def: A smooth fn $f : U \subseteq \mathbb{C} \to \mathbb{C}$



is conformal if at each $p \in U$,

$Df = \left(\frac{\partial f_i}{\partial x_j}\right)$ is a non-zero orient. pres. linear trans which preserves angles (equiv. Df is the comp of a rotation and a dialation)