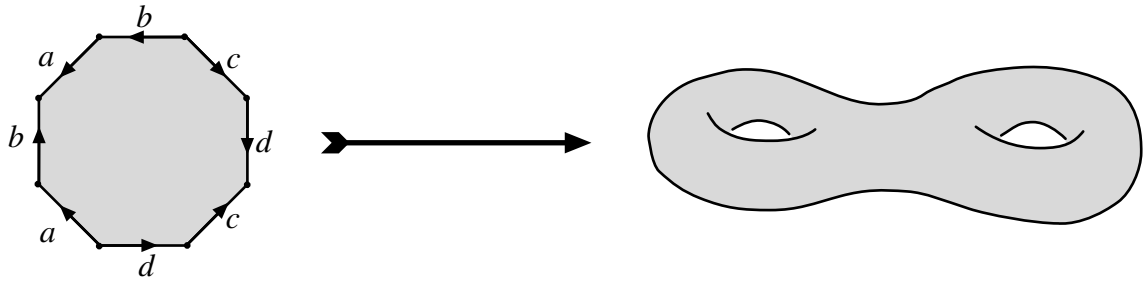
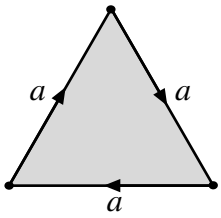
Math 131 Homework #12 (last one!); Due Tuesday April 29

Note on the material covered in class recently: Van Kampen's Theorem, in its various versions, is treated in Chapter 11 in Munkres. The general theory of covering spaces, as I'm discussing this week, is treated in Chapter 13. For more on the fundamental group of a graph, see §14.84. You may also wish to consult Chapter 1 of Allen Hatcher's excellent text "Algebraic Topology" which is what I'm roughly following in class. You can download all of Hatcher's book from: http://www.math.cornell.edu/~hatcher/

1. Draw pictures illustrating how the quotient space defined at left is the surface shown on the right.



2. Let X be the quotient space shown below.



   (a) While X does not embed in $\mathbb{R}^3$, a small neighborhood U of the loop a does. Draw an embedding of U in $\mathbb{R}^3$.

   (b) Compute $\pi_1(X)$.

   (c) Find the universal cover, i.e. a cover $p\colon E \to B$ where E is simply connected. Exhibit an embedding of E in $\mathbb{R}^3$.

3. In each case, find a presentation for the fundamental group of X.

   (a) Let X be the quotient space described as follows. Let $Y = S^1 \times I$; now identify $(z,0)$ with $(z^2,1)$ for each $z \in S^1$ to form X.

   Comment: $G = \pi_1(X)$ is a very interesting group. For instance, it has an element of infinite order x which is conjugate to x^{2n} for any $n \in \mathbb{Z}_+$!

   (b) Let X be the quotient space described as follows. Let $Y = S^1 \times I$; now identify $(z,0)$ with $(w,1)$ if $z^2 = w^3$ to form X.

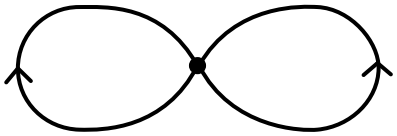
   Comment: $G = \pi_1(X)$ is also interesting; it has the property that it is a proper quotient of itself. That is, there exists a surjective homomorphism $h\colon G \to G$ with non-trivial kernel.

4. Let T be the torus $S^1 \times S^1$; we know that $\pi_1(T) = \mathbb{Z} \times \mathbb{Z}$.

   (a) Let $n \in \mathbb{Z}_+$. Find the cover of T corresponding to the subgroup generated by $(n,0)$.

   (b) Let $n, m \in \mathbb{Z}_+$. Find the cover of T corresponding to the subgroup generated by $\{(n,0),(0,m)\}$.

5. Let X be the figure-8:



   Let x_0 be the marked point at the center. Let $a = [f]$ and $b = [g]$ where where f is a path going around the right loop once and g a path going around the left one once. Then $G = \pi_1(X, x_0)$ is the free group generated by a and b.

   Let $h\colon\ G \to S_3$ be the homomorphism defined by $h(a) = (12)$ and $h(b) = (123)$. Let $H \leq G$ be the stabilizer of 1 under this action on $\{1,2,3\}$, i.e. $g \in H$ if $h(g)(1) = 1$.

   (a) Find a cover $p\colon\ (E, e_0) \to (X, x_0)$ so that $p_*(\pi_1(E, e_0)) = H$.

   (b) Use this to compute $[G : H]$ and find a finite set of generators for H.

   (c) Repeat parts (a) and (b) for the kernel, K, of the homomorphism h.

6. As in the previous problem, let X be the figure-8, and G its fundamental group. For this problem, you may want to look at page 58 of Hatcher for some informative examples.

   (a) Let H be the subgroup of G generated by $\{a, bab, b^3, b^{-1}a^2b^{-1}, ba^{-1}bab^{-1}\}$. Find the corresponding cover of X. What is the index of H in G?

   (b) Let N be the subgroup of G *normally generated* by $\{a^2, b^2, (ab)^4\}$.

      i. Find the cover E of X corresponding to N.

      ii. Identify the finite group $Q = G/N$.

      iii. Describe the action of Q on E by covering transformations so that $E/Q = X$.

      Hint: $[G : N] = 8$.

7. Let X be a path connected, locally path connected, topological space where $\pi_1(X)$ is finite. Show that any continuous map $f\colon\ X \to S^1$ is null-homotopic. Hint: Use the cover $p\colon\ \mathbb{R} \to S^1$.

   Comment: So, for instance, any map $f\colon\ S^2 \to S^1$ is null-homotopic. However, there are maps $f\colon\ X \to Y$ where $\pi_1(X) = 1$ but f is not null-homotopic. The identity map from S^2 to itself is such a map.