Lecture 30: Algebraic and Transcendental Elements.

DF § 13.1–13.2. §5–7 of [R3].

Last time: Field extension $\begin{matrix} K \\ | \\ F \end{matrix} := (F \subset K \text{ fields.})$

$[K:F] := \dim_F K$

Constructions: Given $F \subseteq K$ and $\alpha_1, \ldots, \alpha_n \in K$,

set $F(\alpha_1, \ldots, \alpha_n) :=$ smallest subfield containing $F \cup \{\alpha_1, \ldots, \alpha_n\}$.

Given $p(x) \in F[x]$ irreducible, set $L := F[x]/(p(x))$.

Here $L = F(\theta)$ for $\theta = x + (p(x))$ and $[L:F] = \deg(p)$.

Simple extension: K/F where $K = F(\alpha)$ for some $\alpha \in K$. ← primitive elt.

[Ex: Any $L := F[x]/(p(x))$ as an extension of F.]

Ex: $\mathbb{Q}(\sqrt{2}, \sqrt{5})/\mathbb{Q}$ since $\mathbb{Q}(\sqrt{2}, \sqrt{5}) = \mathbb{Q}(\underbrace{\sqrt{2} + \sqrt{5}}_{\alpha})$

as $\sqrt{2} = \frac{1}{6}(\alpha^3 - 11\alpha)$.

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Two kinds of elements α in K/F:

<u>algebraic</u>: $\exists$ nonzero $f \in F[x]$ with $f(\alpha) = 0$

<u>transcendental</u>: α not a root of any nonzero $f \in F[x]$.

<u>Ex</u>: $\sqrt{2}, \sqrt[3]{5} \in \mathbb{R}$ are algebraic$/\mathbb{Q}$.

$\pi, e \in \mathbb{R}$ are transcendental

If $\alpha \in K$ is algebraic$/F$, consider

$$\tilde{\varphi}: F[x] \longrightarrow K \quad \text{with } f \longmapsto f(\alpha)$$

where $Ker(\tilde{\varphi}) = \{f \mid f(\alpha)\} = 0$. Nonempty and doesn't contain $1 \Rightarrow \exists!$ monic $m_\alpha(x)$ of degree ≥ 1 with $Ker(\tilde{\varphi}) = (m_\alpha(x))$.

We call $m_\alpha(x)$ the <u>minimal polynomial</u> of α.

<u>Thm</u>: Suppose $\alpha \in K$ is algebraic over F. Then $m_\alpha \in F[x]$ is irreducible, and $F(\alpha) \cong F[x]/(m_\alpha(x))$.

<u>Pf</u>: Note $\mathrm{Im}(\tilde{\varphi}) \cong F[x]/(m_\alpha(x))$ since $\ker\tilde{\varphi} = (m_\alpha)$. Now $\mathrm{Im}(\tilde{\varphi})$ is an int domain
$\Rightarrow (m_\alpha)$ prime $\Rightarrow m_\alpha$ irred and (m_α) is maximal
$\Rightarrow \mathrm{Im}(\tilde{\varphi})$ is a field, must be $F(\alpha)$. ▨

<u>Thm</u>: Suppose $K = F(\alpha)$ with $[K:F] = n < \infty$. Then α is algebraic over F and $\deg(m_\alpha) = n$.

<u>Pf</u>: As $\dim_F K = n$, the elts $1, \alpha, \ldots, \alpha^n$ are linearly dependent, say $a_0 \cdot 1 + a_1\alpha + \cdots + a_n\alpha^n = 0$ for some $\alpha_i \in F$ not all 0. Then α is a root of $a_0 + a_1x + \cdots + a_nx^n \in F[x]$ and so is alg$/F$.
Hence $\deg m_\alpha \leq n$ and must be $= n$ since $F(\alpha) \cong F[x]/(m_\alpha)$ which as degree $= \deg m_\alpha / F$. ▨

Ex: $\mathbb{Q}(\sqrt{2}, \sqrt{5})$ has $\mathbb{Q}$-basis $1, \sqrt{2}, \sqrt{5}, \sqrt{10}$

We compute

$$\begin{aligned}
\alpha &= \sqrt{2} + \sqrt{5} \\
\alpha^2 &= 7 + 2\sqrt{10} \\
\alpha^3 &= 17\sqrt{2} + 11\sqrt{5} \\
\alpha^4 &= 89 + 28\sqrt{10}
\end{aligned}$$

$\Rightarrow \alpha^4 - 14\alpha^2 + 9 = 0$. In fact $m_\alpha = x^4 - 14\alpha^2 + 9$.

What about simple extensions with $[F(\alpha) : F] = \infty$?

Ex: $F(x) := \mathrm{Frac}(F[x])$ field of rat'l fns.

Any simple $F(\alpha)/F$ of ∞ degree is isom to $F(x)$:

$$\phi : F(x) \longrightarrow F(\alpha) \quad \text{with} \quad \frac{p(x)}{q(x)} \longmapsto \frac{p(\alpha)}{q(\alpha)}$$

makes sense as $q(\alpha) \neq 0$ if $q \neq 0$ in $F[x]$.

ϕ is surjective, $1 \notin \ker \phi \Rightarrow \ker \phi = \{0\}$

$\Rightarrow \phi$ is an isomorphism. $\square$

Cor: $\mathbb{Q}(\pi), \mathbb{Q}(e), \mathbb{Q}(\log 2)$ are all isomorphic fields.

K/F is algebraic if <u>every</u> $\alpha \in K$ is algebraic over F.

<u>Ex</u>: K/F with $[K:F] < \infty$ by previous thms, since $[F(\alpha):F] \leq [K:F]$.

<u>Ex</u>: $\mathbb{Q}^{alg} := \{\alpha \in \mathbb{C} \mid \alpha \text{ algebraic over } \mathbb{Q}\}$

<u>Q</u>: Why is this a subfield?

<u>A</u>. <u>Prop</u>: If L/F is an extension with $\alpha, \beta \in L$ algebraic over F, then $\alpha+\beta$, $\alpha\beta$, $-\alpha$, α^{-1} are all alg. over F.

$$\begin{array}{c} F(\alpha,\beta) \\ | \\ F(\alpha) \\ | \\ F \end{array}$$

<u>Pf</u>. Know $[F(\alpha):F] = \deg m_{F,\alpha}$. α is alg$/F$. Now β is alg over $F(\alpha)$ and $[F(\alpha,\beta):F(\alpha)] = \deg m_{F(\alpha),\beta}$. Thus $[F(\alpha,\beta):F]$

$= [F(\alpha,\beta):F(\alpha)][F(\alpha):F] < \infty$

$\Rightarrow$ every elt of $F(\alpha,\beta)$ is alg$/F$. ▨

<u>Q</u>: What is $[\mathbb{Q}^{alg}:\mathbb{Q}]$?

<u>A</u>: Infinite since $\mathbb{Q}^{alg} \supseteq \mathbb{Q}(\sqrt[n]{2})$ and $[\mathbb{Q}(\sqrt[n]{2}):\mathbb{Q}] = n$ as $x^n - 2$ is irred by Eisenstein.