Lecture 28: More on canonical forms for linear transformations.

§68 of [RZ]
§12.2-12.3 of [DF]

Previously... F field, V an F-vector space

$T: V \to V$ linear operator.

$V_T :=$ The $F[x]$ module with additive group $(V,+)$ and $f(x)\cdot v = f(T)(v)$.

$\dim_F V < \infty \iff V_T$ is a finitely gen torsion mod.

$$\iff V_T \underset{\text{as } F[x] \text{ mod}}{\cong} F[x]/(f_1) \oplus F[x]/(f_2) \oplus \cdots \oplus F[x]/(f_m)$$

where f_i are monic, non zero, and all $f_i \mid f_{i+1}$.

Silly def: minimal poly of T is $m_T(x) := f_m$ from above.

Consider $J_T := \{ f \in F[x] \mid f(T) = 0 \}$ — 0 lin op, sends all of V to 0_V.

Concretely, given $n\times n$ matrix A with F entries, consider

$$J_A := \{ f \in F[x] \mid f(A) = 0 \}$$

Ex: $A = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$, then $x^2 - 3x + 2 \in J_A$ since
$A^2 - 3A + 2\cdot I = 0$

J_T is an ideal, must be principal.

Better def: the min poly of T is the unique monic $m_T \in F[x]$ with $J_T = (m_T(x))$.

Equivalently, $m_T(x)$ is the nonzero monic elt of $F[x]$ of lowest degree with $m_T(T) = 0$.

Connection: $J_T = \mathrm{Ann}(V_T) = \{ f \in F[x] \mid f \cdot V_T = 0 \}$

$$\Updownarrow$$

$$f(T) \cdot v = 0 \quad \forall v \in V.$$

$= (f_m)$ if V_T has invariant factor decomp. as above.

[Need to relate to the char. polynomial.]

Thm: The characteristic poly $\mathrm{char}_T(x) := \det(x \cdot I - T)$ is divisible by $m_T(x)$. Moreover, they have the same roots.

Ex: $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ has $m_A = x - 1$ but $\mathrm{char}_A = (x-1)^2$.

Pf: Choose a basis β where $[T]_\beta$ is in rational canonical form, with diagonal blocks $\underbrace{C_{f_1}, C_{f_2}, \ldots, C_{f_\ell}}_{\text{companion matrices.}}$ where f_i are monic with $f_i \mid f_{i+1}$. Now $\det[T_\beta]$

$$= \prod \det(C_{f_i}) \underset{\substack{\uparrow \\ \text{Check!}}}{=} \prod f_i$$

$$xI - C_{f_i} = \begin{pmatrix} x & 0 & 0 & & & c_0 \\ -1 & x & 0 & & 0 & c_1 \\ 0 & -1 & x & & & \vdots \\ & & -1 & \ddots & & \\ & 0 & & & x & \\ & & & & -1 & x + c_{d-1} \end{pmatrix}$$

for $f_i = x^d + c_{d-1}x^{d-1} + \cdots + c_0$

Since $m_T = f_\ell$, get $m_T \mid \mathrm{char}_T$. If $\mathrm{char}_T(c)$ is 0, then $f_i(c) = 0$ for some $i \Rightarrow m_T(c) = 0$ as $f_i \mid m_T$. ▨

<u>Cor</u>: (Cayley-Hamilton) T lin op on f.d. V. Then $\mathrm{char}_T(T) = 0$.

Suppose $m_T(x) = (x - c_1)(x - c_2)\cdots(x - c_\ell)$ for $c_i \in F$ [Always true for $F = \mathbb{C}$.] Then [primary decomp!]

$$V_T \cong F[x]/(x - c_1)^{k_1} \oplus \cdots \oplus F[x]/(x - c_\ell)^{k_\ell} \quad ✪$$

as $F[x]$-modules. Now

$$F[x]/(x-c)^k \text{ has } F\text{-basis } \underset{e_1}{(\bar{x} - c)^{k-1}}, \underset{e_2}{(\bar{x} - c)^{k-2}}, \ldots$$

$$e_{k-1} = \bar{x} - c \text{ and } e_k = 1.$$

(To see this relate to known F-basis $1, \bar{x}, \bar{x}^2, \ldots, \bar{x}^{k-1}$).

Note: $x \cdot e_i = \bar{x}(\bar{x}-c)^i = (\bar{x}-c+c)(\bar{x}-c)^i$

$$= \underbrace{(\bar{x}-c)^{i+1}}_{= e_{i-1} \text{ if } i>1 \text{ else } 0.} + c\, e_i$$

So matrix of mult by x on $F[x]/_{(x-c)^k}$ is

$$\begin{pmatrix} c & 1 & 0 & 0 & & 0 \\ & c & 1 & 0 & & 0 \\ & & c & 1 & & 0 \\ & & & \ddots & \ddots & 0 \\ & 0 & & & \ddots & 1 \\ & & & & & c \end{pmatrix} =: J_{c,k}$$

Using such bases and ★, we find a matrix for T that is in Jordan canonical form, meaning block-diagonal with various $J_{c,k}$ blocks:

$$\begin{pmatrix} \boxed{\begin{matrix} 2 & 1 \\ 0 & 2 \end{matrix}} & & & & & & & & \\ & \boxed{2} & & & & & & 0 & \\ & & \boxed{\begin{matrix} 3 & 1 & \\ & 3 & 1 \\ & & 3 \end{matrix}} & & \\ & 0 & & \boxed{\begin{matrix} -1 & 1 \\ 0 & -1 \end{matrix}} & \\ & & & & \boxed{7} \end{pmatrix}$$

$$\boxed{\begin{matrix} m_T = \\ (x-2)^2(x-3)^3(x+1)^2(x-7) \end{matrix}}$$

This is unique up to ordering of blocks.

[If time remains, contrast rat'l and Jordan canonical forms...]