Lecture 27: Canonical forms for linear maps

§68 of [R2]
§12.2 - 12.3 of [DF].

Last time: R a PID.

M a finitely generated R-module

Then $\exists\, t \geq 0$ and $a_i \in R$ with

$$M \cong R/(a_1) \oplus R/(a_2) \oplus \cdots \oplus R/(a_t)$$

and $R \supsetneq (a_1) \supseteq (a_2) \supseteq \cdots \supseteq (a_t)$.

Invariant factor decomposition

Also, $\exists\, r, u \geq 0$ and p_i prime in R; with $k_i \geq 1$ and

$$M \cong R^r \oplus R/(p_1^{k_1}) \oplus \cdots \oplus R/(p_u^{k_u})$$

Primary decomposition

Here, t, r, u are unique, a_i unique up to associates, $p_i^{k_i}$ unique up to order and assoc.

Cor: Classification of finitely generated abelian gps.

Pf: Take $R = \mathbb{Z}$.

Today: application when $R = F[x]$, F a field.

Setting: V an F-vector space.

$T : V \to V$ linear operator = linear trans from a vector space to itself.

Goal: Find a basis β for V where the matrix $[T]_\beta$ for T is simple (e.g. diagonal).

<u>Construction</u>: Give V the structure of a $F[x]$-module by $f \cdot v := f(T)v$ for $f \in F[x]$, $v \in V$.

Here, recall can add $T, S \in \mathrm{Hom}_F(V,V)$ and also compose them (as functions) so e.g.

$$\underbrace{U = T^3}_{T \circ T \circ T} + 2T + 3$$ is also a linear operation where 3 is really $3 \cdot \mathrm{Id}_V$.

<u>Note</u>: Can compute $[U]_\beta$ from $[T]_\beta$ by adding / mult matrices in the nat'l way.

Write V_T for this $F[x]$-module. [Conversely, given an $F[x]$-module W, can view W as an F-vectorspace with a linear operator $U(w) := x \cdot w$. (check!)]

Submodules of V_T $\longleftrightarrow$ T-invariant subspaces $W \subseteq V$, i.e. $T(W) \subseteq W$.

V_T and V_U are isomorphic as $F[x]$ modules $\longleftrightarrow$ T and U are <u>similar</u>, i.e. $\exists$ a lin. op S on V with $U = S \circ T \circ S^{-1}$.

V_T finitely gen and torsion $\longleftrightarrow$ V finite dim'l

Reason: $(\longleftarrow)$ V finite-dim'l $\Leftrightarrow$ f.g. as an F-mod $\Rightarrow$ f.g. as an $F[x]$ mod. V_T can't have a $F[x]$ summand as $\dim_F F[x] = \infty$. So

$$V_T \cong F[x]/(f_1) \oplus \cdots \oplus F[x]/(f_m) \quad (\star)$$

Suppose $f(x)$ is monic $= x^d + b_{d-1}x^{d-1} + \cdots + b_1 x + b_0$

Then $F[x]/(f)$ has basis $\underset{e_1}{\overset{\shortparallel}{\bar{1}}}, \underset{e_2}{\overset{\shortparallel}{\bar{x}}}, \ldots, \underset{e_d}{\overset{\shortparallel}{\bar{x}^{d-1}}}$ where

$\bar{1} = 1 + (f)$, $\bar{x} = x + (f)$, etc. since $\bar{x}^d = -(b_{d-1}\bar{x}^{d-1} + \cdots + b_0)$

Mult by x on $F[x]/(f)$ has matrix

$$C_f := \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & -b_0 \\ 1 & & & & 0 & -b_1 \\ 0 & 1 & & & 0 & -b_2 \\ \vdots & 0 & \ddots & & \vdots & \vdots \\ & & & \ddots & & \\ & & & & 1 & -b_{k-1} \end{pmatrix}$$

I_{d-1}

[called the companion matrix for f]

A matrix is in rational canonical form

when it is block-diagonal

$$\begin{pmatrix} B_1 & 0 & & \\ & B_2 & & 0 \\ & & \ddots & \\ 0 & & & B_m \end{pmatrix},$$

where each $B_i = C_{f_i}$

for a monic $f_i \in F[x]$

<u>and</u> $f_i \mid f_{i+1}$ for all i.

<u>Thm</u>: Every linear op T on a finite-dim'l F-vector space V has a basis β where $[T]_\beta$ is in rational canonical form.

<u>Pf</u>: Use the invariant factor form of V_T. $\blacksquare$

<u>Thm</u>: Two linear ops on V are similar $\Longleftrightarrow$ they have the same rational canonical form.

<u>Pf</u>: Uniqueness part of Classification of $F[x]$ modules.

<u>Silly def</u>: The <u>minimal polynonomial</u> m_T of T is that corresponding the final block in the rational canonical form.

Better def:

M an R-module. The annihilator of M is the ideal
$$\text{Ann}(M) := \{ r \in R \mid r \cdot M = 0 \iff r \cdot m = 0 \ \forall m \in M \}$$

Ex: $R = \mathbb{Z}$ $\quad \text{Ann}(\mathbb{Z}/2 \oplus \mathbb{Z}/3) = (6)$

$\text{Ann}(\mathbb{Z}^2) = (0)$.

Ex: $\text{Ann} \neq \{0\} \Rightarrow M$ is torsion

Ex: R a PID, M finitely gen

Then $\text{Ann}(M) =$ last ideal in the invariant factor decomposition.

Given $T : V \to V$ a linear op of an F-vector space,

the min poly is the monic generator of $\text{Ann}(V_T)$.