Lecture 2: Isomorphism Theorems for Groups

Def: A function $\phi : G \to H$ between groups is a <u>homomorphism</u> when $\phi(xy) = \phi(x) \cdot \phi(y)$ for all $x, y \in G$.

Ex: $G = GL_n \mathbb{F}$ $\quad \phi = \det$.
$H = \mathbb{F}^\times = (\mathbb{F} \setminus \{0\}, \times)$

An <u>isomorphism</u> of groups is a bijective homomorphism.

Ex: S_2 and C_2 are isomorphic. **Ex**: $GL_4 \mathbb{F}_2 \cong A_8$.

Note: $\ker(\phi) = \{g \in G \mid \phi(g) = e_H\}$ and $\operatorname{im}(\phi) = \{\phi(g) \mid g \in G\}$ are subgroups of G and H respectively.

For $H \leq G$ and $x \in G$, consider the <u>conjugate subgp</u>:

$$xHx^{-1} := \{xhx^{-1} \mid h \in H\}$$

H is <u>normal</u> when $H = xHx^{-1}$ for all $x \in G$. Write $H \trianglelefteq G$.

Ex: For abelian G, any H is normal.

Ex: For any homom. ϕ, $\ker(\phi)$ is normal: if $\phi(g) = 1$ then $\phi(xgx^{-1}) = \phi(x)\phi(g)\phi(x^{-1}) = \phi(x) \cdot (\phi(x))^{-1} = 1$

For $H \trianglelefteq G$, the left-cosets G/H are a group with operation $(xH)\cdot(yH) = (xy)\cdot H$. There is a surjective homomorphism $\pi : G \longrightarrow G/H$, $g \longmapsto gH$, whose kernel is H.

[G/H is the quotient group; π the quotient homom.]

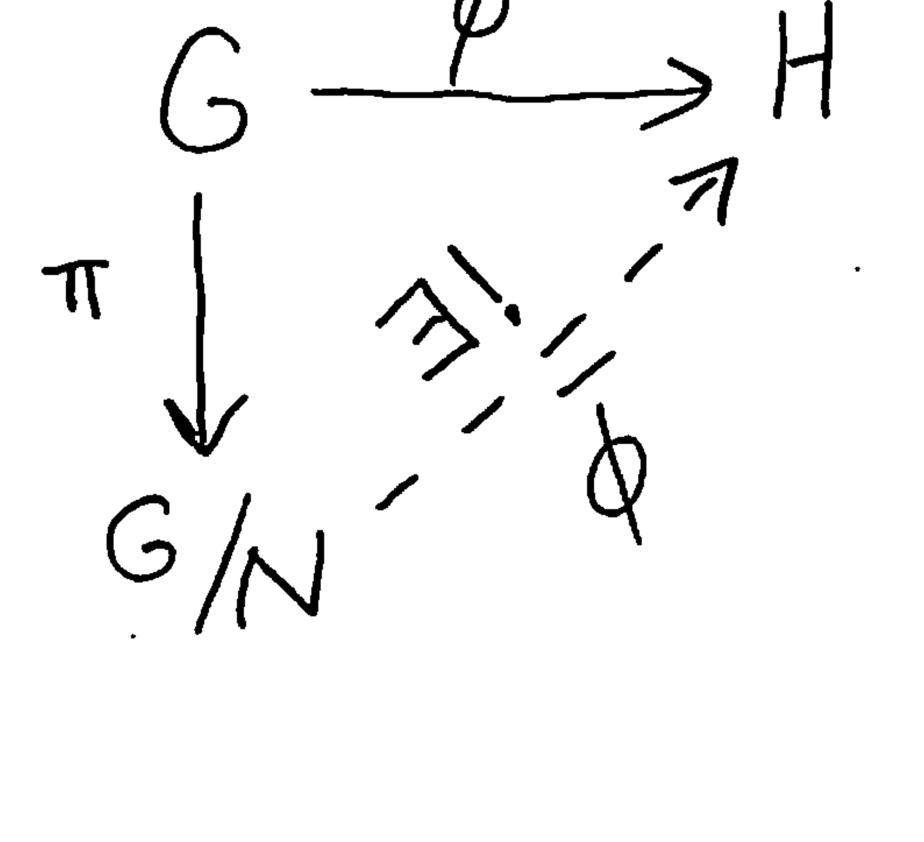
[1st Isomorphism]
Thm: For a homomorphism $\phi : G \to H$, have $\operatorname{Ker}\phi \trianglelefteq G$ and $G/\operatorname{Ker}(\phi) \cong \operatorname{im}(\phi)$ as groups.

Cor: $[G : \operatorname{Ker}(\phi)] = |\operatorname{im}(\phi)|$

Cor: ϕ is injective (1-1) $\Longleftrightarrow \operatorname{Ker}(\phi) = \{e_G\}$

[Before proving the theorem, need to understand how to const. homom. of G/N.]

Homomorphism Lemma: Suppose $N \trianglelefteq G$ with $\pi : G \to G/N$ the quot. hom. If $\phi : G \to H$ is a hom. with $\phi(N) = \{e\}$, then there exists a unique hom. $\bar{\phi} : G/N \to H$ with $\bar{\phi} \circ \pi = \phi$.



Say: ϕ factors through G/N

Say: diagram commutes.

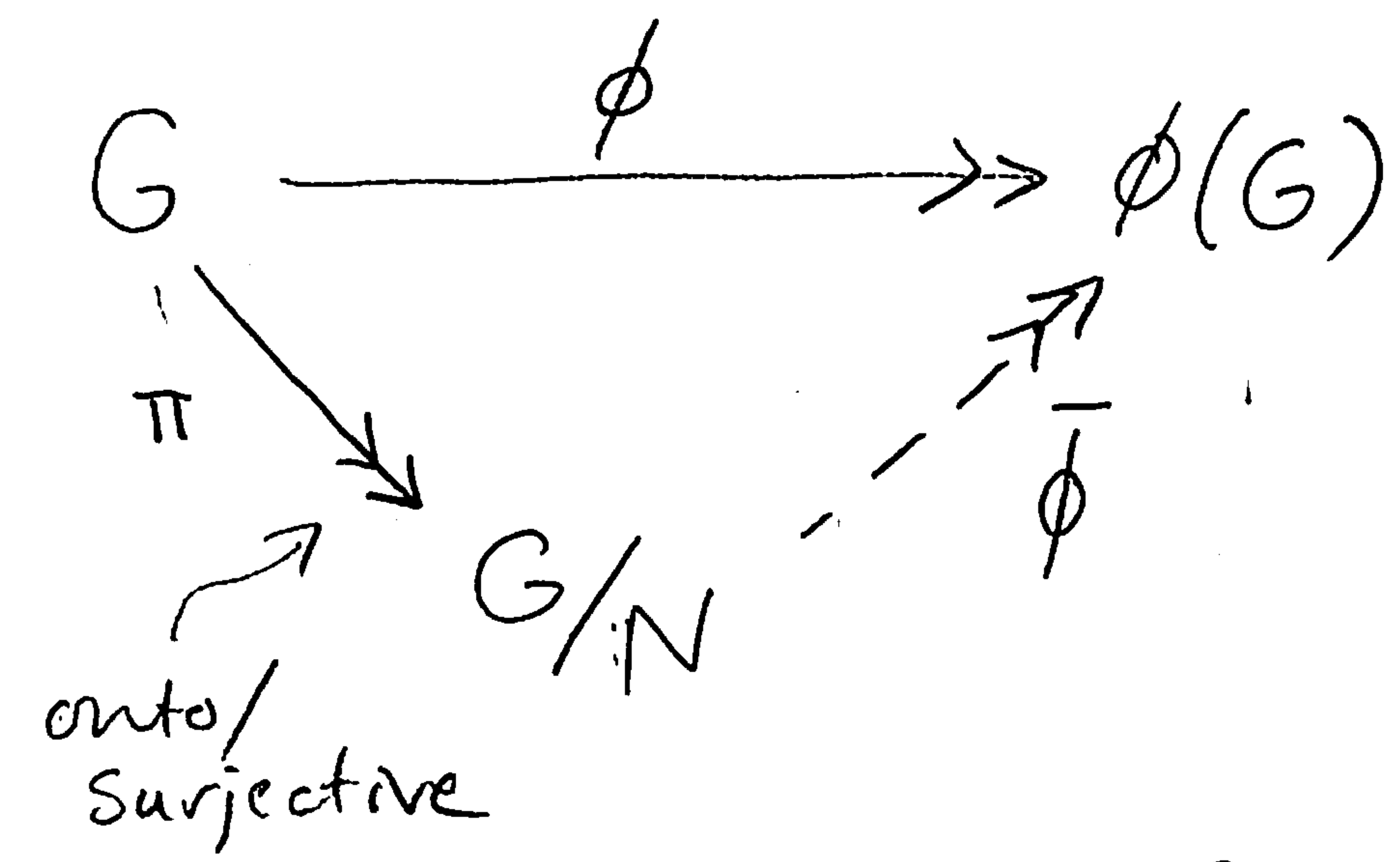
<u>Proof idea</u>: As $xN = \pi(x)$, only possibility is

$\bar{\phi}(xN) := \phi(x)$. Well-defined: If $x' = xn$, then $\phi(x') = \phi(x)\phi(n) = \phi(x)$ by hyp.

Easy exercise to check this is a homom. □

<u>Proof of Thm</u>:

Set $N = \ker(\phi)$



Now $\ker \bar{\phi} = \{ xN \mid \bar{\phi}(xN) = e \} = \{N\} = \{ e_{G/N} \}$.
(where $\bar{\phi}(xN) = \phi(x)$)

So $\bar{\phi}$ is injective, hence an isomorphism. □

$A, B \leq G$. Then $A \cap B$ is also a subgp.
But $A \cup B$, $AB = \{ ab \mid a \in A, b \in B \}$ and BA are often not subgps.

<u>Def</u>: For $H \leq G$, the <u>normalizer</u> is

$$N_G(H) = \{ g \in G \mid gHg^{-1} = H \} \leq G.$$

<u>Notes</u>: $H \trianglelefteq N_G(H)$ and $H \trianglelefteq G \iff N_G(H) = G$.

[Second (Diamond) Isomorphism Theorem]

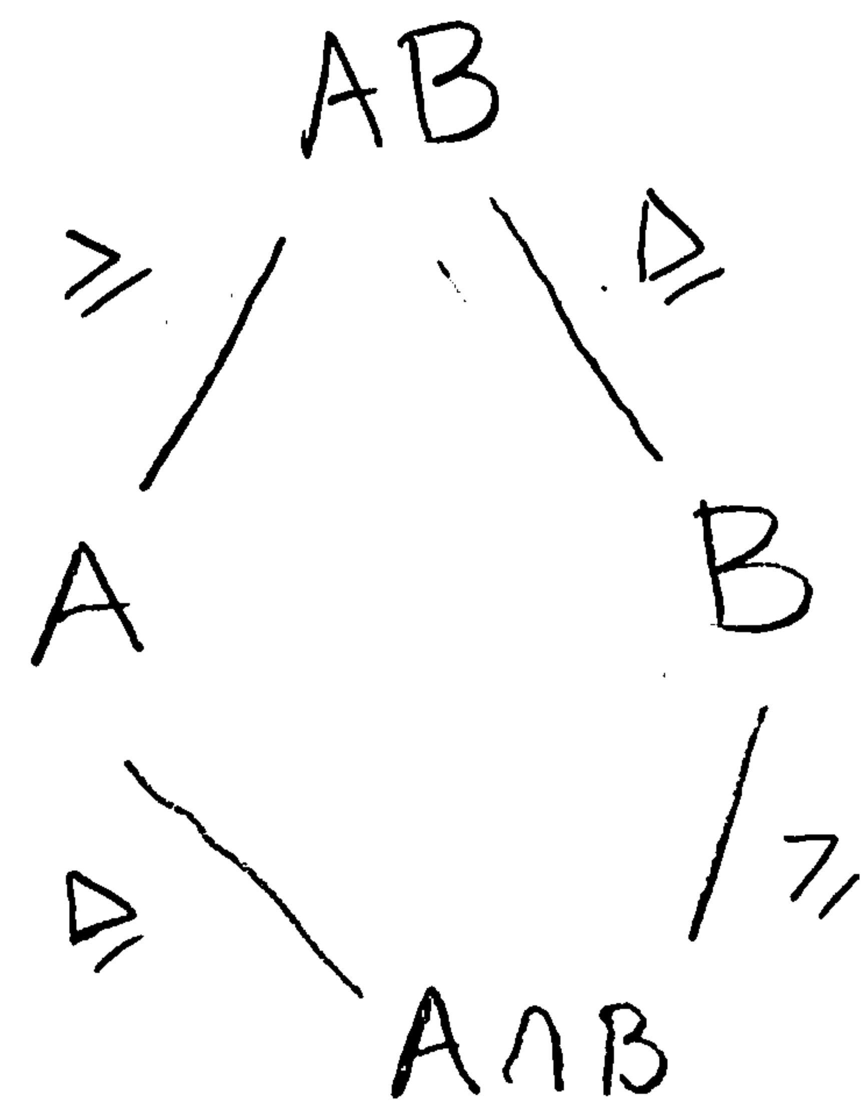
<u>Thm</u> Suppose $A, B \leq G$ and $A \leq N_G(B)$. Then

1) $AB = BA$ is a subgp of G.

2) $B \trianglelefteq AB$

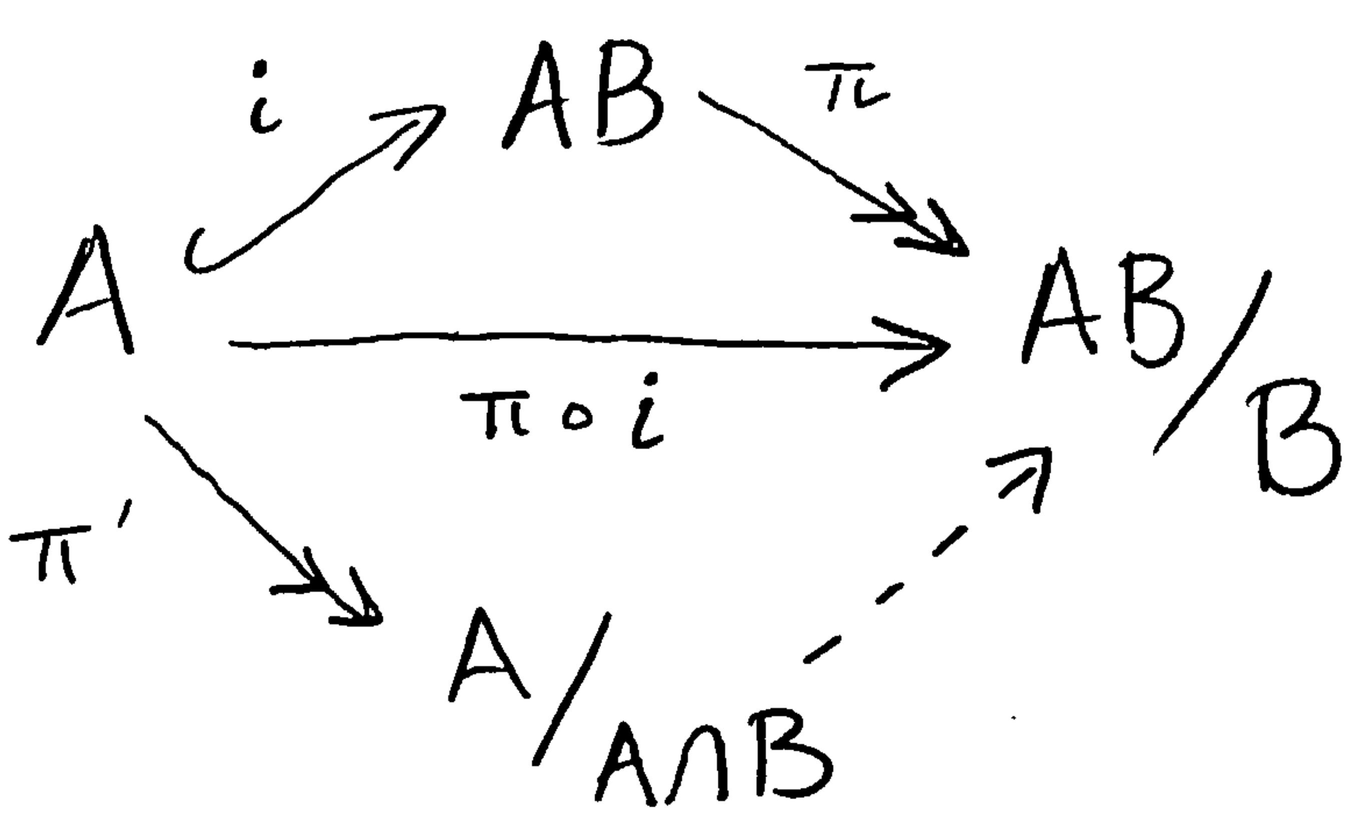
3) $A \cap B \trianglelefteq A$

4) $A/(A \cap B) \cong AB/B$



<u>Pf</u>: See text / Rezk's notes.

Key:



[Third Isom Thm]

<u>Thm</u>: Suppose $H \trianglelefteq G$ and $K \trianglelefteq G$. If $H \leq K$, then

① $K/H \trianglelefteq G/H$ ② $(G/H)/(K/H) \cong G/K$

<u>Key idea</u>: