Lecture 12: Semi-direct products and composition series.

§ 46–48 of [RI]
§5.5 and 3.4 of [DF]

Last time: G is an extension of K by H

when $H \overset{i}{\hookrightarrow} G \overset{p}{\twoheadrightarrow} K$ with $i(H) = \mathrm{Ker}(p)$.

This splits when $\exists s: K \to G$ with $p \circ s = id_K$.

($\Rightarrow s(K) \cong K$). Not split: $C_2 \hookrightarrow C_4 \twoheadrightarrow C_2$

Q: Given K, H classify all extensions of K by H up to isomorphism.

[Can do using group cohomology, but hard in practice.]

[Relatively simple for split extensions ...]

Suppose G is a split extension of K by H. To reduce notation, identify H and $i(H)$ as well as K and $s(K)$. Define a hom. $\alpha: K \to \mathrm{Aut}(H)$ by $\alpha(k) = \mathrm{conj}_k\big|_H$.

Fact: α determines G; any such α appears in some split extension.

Semi-direct product: Given groups H, K, and $\alpha: K \to \mathrm{Aut}(H)$, define $H \rtimes_\alpha K$ to be the group whose elts are $H \times K$ with operation $(h_1, k_1)\cdot(h_2, k_2) = (h_1 \underbrace{\alpha(k_1)}_{\in \mathrm{Aut}(H)}(h_2),\ k_1k_2)$

Ex: α trivial, i.e. $\alpha(k) = id_K$. Then $H \rtimes_\alpha K = H \times K$.

Ex: $K = C_2 = \langle s \mid s^2 \rangle$. $H = C_n = \langle r \mid r^n \rangle$

$\alpha(s) \in \mathrm{Aut}(H)$ be $h \mapsto h^{-1}$. (an auto. since H abelian)

Then $H \rtimes_\alpha K \cong D_{2n}$.

Note: Any $G = H \rtimes_\alpha K$ contains $H' = \{(h,e) \mid h \in H\}$ a normal subgp $\cong H$ and $K' = \{(e,k) \mid k \in K\}$ a subgp $\cong K$. Can check that G is split ext of K by H with assoc. hom α.

Motivation: If $N \lhd G$ then $(n_1 g_1)\cdot(n_2 g_2)$
$= n_1(g_1 n_2 g_1^{-1})\, g_1 g_2 = n_1\big(\mathrm{conj}_{g_1}(n_2)\big)(g_1 g_2)$.

See §5.5 of [DF] or §46 of [R1] for more.

[On to composition series, using sequences of extensions to build very complicated groups.]

A group G is simple when its only normal subgps are $\{e\}$ and G; by convention the trivial gp is not simple.

Ex: C_p for p prime.

Ex: A_n for $n \geq 5$.

Ex: $PSL_2\mathbb{F}_p := SL_2\mathbb{F}_p / \{\pm 1\}$ for $p \geq 5$.

Classification of finite simple groups: 18 infinite families + 26 others. (including the "monster" M with $|M| \approx 8 \times 10^{53}$)

A composition series for G is a finite chain of subgps $\{e\} = M_0 \trianglelefteq M_1 \trianglelefteq M_2 \trianglelefteq \cdots \trianglelefteq M_{n-1} \trianglelefteq M_n = G$ where each M_k / M_{k-1} is simple. ← composition factors.

Ex: $G = C_6 = \langle a \mid a^6 \rangle$

$$\{e\} \trianglelefteq \underset{\substack{\cong \\ C_3}}{\langle a^2 \rangle} \trianglelefteq G \quad \text{and} \quad \{e\} \trianglelefteq \underset{\substack{\cong \\ C_2}}{\langle a^3 \rangle} \trianglelefteq G$$

<u>Ex</u>: $D_{12} = \langle s, r \mid s^2, r^6, (sr)^2 \rangle$

$\{e\} \trianglelefteq \langle r^2 \rangle \trianglelefteq \langle r \rangle \trianglelefteq D_{12}$.

Composition factors (C_3, C_2, C_2).

<u>Ex</u>. S_n for $n \geq 5$. $\{e\} \trianglelefteq A_n \trianglelefteq S_n$.

<u>Note</u>: Any finite G has a comp. series (induct on $|G|$).

<u>Jordan-Holder Thm</u>: The composition factors of a finite G are unique up to permutation. That is, if $\{e\} = M_0 \trianglelefteq \cdots \trianglelefteq M_r = G$ and $\{e\} = N_0 \trianglelefteq \cdots \trianglelefteq N_s = G$ are composition series then $r = s$ and $\exists\, \sigma \in S_r$ with

$$M_k / M_{k-1} \cong N_{\sigma(k)} / N_{\sigma(k-1)}.$$

<u>Pf</u>: See § 48 of Rezk.

<u>Moral</u>: In theory, if we understand simple groups and group extensions well enough we can understand all finite gps.