Math 500: Final the Ultimate

Date/Time/Location: Thursday, May 15 from 1:30am–4:30pm in our usual classroom.

Office hours: My remaining office hours are:

- Wednesday, May 7 from 1:30–2:30pm.
- Thursday, May 8 from 2:00–3:00pm.
- Friday, May 9 from 11:30am–1pm.
- Wednesday, May 14 from 11:30am–1pm (online).

Test format: There will be roughly twice as many questions as on one of the midterms. Most of the questions will be of similar difficulty to those on the second midterm, but a few many be more involved and/or difficult. I will not ask you to repeat proofs of theorems covered in class or in one of the texts.

Cheat sheet: The text will be closed book, but you may bring **three** sheets of standard US letter or A4 paper to the exam, on which you have written, printed, or copied anything that you think will be helpful. You can use both sides of each sheet, but must be able to read it without a magnifying glass or suchlike aide.

Material covered: The final will be comprehensive. The material on fields and Galois theory will make up about half the exam, with groups and rings each being about one quarter of the exam. Fair game is anything from:

- Any lecture.
- Part 1 of Rezk's notes.
- Part 2 of Rezk's notes, except for Sections 10, 24, 34, 47, 53–55.
- The following parts of Dummit and Foote: Chapters 1–6, with the exception of 6.2, Sections 7.1–7.5, Chapter 8, Chapter 9 except for the part of Section 9.6 after Corollary 22, Sections 10.1–10.3, Chapter 12, Chapter 13 except for 13.3 and 13.5, and Sections 14.1–14.3 and 14.5–14.7.

By far the most important material is that which appeared in lecture or on the homework.

Study resources: The course webpage, which is `http://dunfield.info/500` contains scanned lecture notes and solutions to the HW problems.

Homework problems for material after HW 11: Four problems about the material covered in the last three lectures is on the back of this sheet. Solutions will be posted.

Practice exam: A practice exam has been posted. You may also find reviewing the practice and actual exams for the first two midterms to be helpful. You may find additional useful practice questions on old algebra comp exams.

`https://math.illinois.edu/academics/graduate-program/coursework-and-exams/study-comp-exams`

Here is what would have been on a HW 12, if there was one. That is, here are some homework problems corresponding to the last three lectures. These are not to be turned in, and solutions can be found on the course webpage.

1. Determine the Galois groups of the following cubic polynomials:
   (a) $f(x) = x^3 - 2x + 4$
   (b) $g(x) = x^3 - x + 1$
   (c) $h(x) = x^3 + x^2 - 2x - 1$
2. Let $f(x)$ be an irreducible polynomial of degree 4 in $\mathbb{Q}[x]$ with discriminant D. Let K denote the splitting field of $f(x)$, viewed as a subfield of the complex numbers $\mathbb{C}$.
   (a) Prove that $\mathbb{Q}(\sqrt{D}) \subset K$.
   (b) Let τ denote complex conjugation and τ_K its restriction to K. Prove that τ_K is an element of $\text{Gal}(K/\mathbb{Q})$ of order 1 or 2 depending on whether every element of K is real or not.
   (c) Prove that if $D < 0$ then $\text{Gal}(K/\mathbb{Q})$ cannot be isomorphic to C_4.
   (d) Prove generally that $\mathbb{Q}(\sqrt{E})$ for squarefree $E < 0$ is not a subfield of any Galois $L/\mathbb{Q}$ where $\text{Gal}(L/\mathbb{Q}) \cong C_4$.
3. Let $K = \mathbb{Q}(\zeta_{2^n})$ for $n \geq 3$. Show that K has exactly three subfields L with $[K : L] = 2$ and find primitive elements for each one. Which of these subfields are Galois over $\mathbb{Q}$?

   Hint: From Section 14.5 of [DF], we know $K/\mathbb{Q}$ is Galois with group $(\mathbb{Z}/2^n\mathbb{Z})^\times$; moreover, you can determine the structure of $(\mathbb{Z}/2^n\mathbb{Z})^\times$ from Exercises 22 and 23 in Section 2.3 of [DF].
4. Consider $f(x) = (x^3 - 2)(x^3 - 3)$ in $\mathbb{Q}[x]$.
   (a) Determine the Galois group of $f(x)$ over $\mathbb{Q}$. That is, if K is the splitting field of f, compute $\text{Gal}(K/\mathbb{Q})$. Note: You may assume that $\mathbb{Q}(\sqrt[3]{2})$ and $\mathbb{Q}(\sqrt[3]{3})$ are distinct subfields of K. (This can be verified by checking that $\sqrt[3]{3} - \sqrt[3]{2}$ is a root of $x^9 - 3x^6 + 165x^3 - 1$ and that the latter polynomial is irreducible.)
   (b) Find all subfields of K that contain $\mathbb{Q}(\zeta)$, where ζ is a primitive 3^{rd} root of unity.

Credit: Problems 1, 2, and 4 from Dummit and Foote.