

Math 500: //DRAFT ONLY// HW 11 due Friday, May 2, 2025.

- Let F be a field of characteristic $\neq 2$.
 - Suppose $K = F(\sqrt{D_1}, \sqrt{D_2})$ for D_1 and D_2 in F where none of D_1 , D_2 , or D_1D_2 is a square in F . Prove that K/F is a Galois extension with $\text{Gal}(K/F)$ isomorphic to the Klein 4-group.
 - Conversely, suppose K/F is a Galois extension with $\text{Gal}(K/F)$ isomorphic to the Klein 4-group. Prove that $K = F(\sqrt{D_1}, \sqrt{D_2})$ for D_1 and D_2 in F where none of D_1 , D_2 , or D_1D_2 is a square in F .
- Suppose K is the splitting field over \mathbb{Q} of a cubic polynomial $f(x) \in \mathbb{Q}[x]$. Show that if $\text{Gal}(K/\mathbb{Q})$ is the cyclic group of order 3, then all the roots of f are real. Hint: by calculus, f must have at least one real root.
- Suppose $\mathbb{Q}(\alpha)/\mathbb{Q}$ is algebraic and let L be the splitting field of $m_{\alpha, \mathbb{Q}}(x)$. If p is a prime dividing the order of $\text{Gal}(L/\mathbb{Q})$, show that there is a subfield F of L with $[L : F] = p$ and $L = F(\alpha)$.
- Consider $K = \mathbb{Q}(\sqrt[8]{2}, i)$ from Example 3 of Section 14.2 of [DF], and let $F_1 = \mathbb{Q}(i)$, $F_2 = \mathbb{Q}(\sqrt{2})$, $F_3 = \mathbb{Q}(\sqrt{-2})$. Prove that $\text{Gal}(K/F_1) \cong \mathbb{Z}/8$, $\text{Gal}(K/F_2) \cong D_8$, $\text{Gal}(K/F_3) \cong Q_8$. (Note: in particular, you need to show all these extensions are Galois extensions.)
- Let $L \subseteq \mathbb{C}$ be a subfield such that L/\mathbb{Q} be a finite Galois extension of degree 3, and let $\omega := e^{2\pi i/3} \in \mathbb{C}$. Show that if $\omega \notin L$, then $L(\omega)/\mathbb{Q}$ is a Galois extension with $\text{Gal}(L(\omega)/\mathbb{Q}) \approx \mathbb{Z}/3 \times \mathbb{Z}/2 \approx \mathbb{Z}/6$.
- Suppose $f \in \mathbb{Q}[x]$ is an irreducible degree 4 polynomial and L a splitting field for f . Suppose further than $G = \text{Gal}(L/\mathbb{Q}) \approx S_4$. Let θ be a root of f and set $K = \mathbb{Q}(\theta)$. Prove that K is an extension of \mathbb{Q} of degree 4 which has no proper subfields. Are there any Galois extensions of degree 4 over \mathbb{Q} with no proper subfields?
- Find a primitive generator for $\mathbb{Q}(\sqrt{2}, \sqrt{3}, \sqrt{5})$ over \mathbb{Q} . Be sure to justify your answer.
- Let K/F be a Galois extension and $\alpha \in K$. Define the *norm* of α from K to F to be

$$N_{K/F}(\alpha) = \prod_{\sigma \in \text{Gal}(K/F)} \sigma(\alpha)$$

- Prove that $N_{K/F}(\alpha) \in F$.
- Prove that $N_{K/F}(\alpha\beta) = N_{K/F}(\alpha)N_{K/F}(\beta)$. Thus, the norm gives a group homomorphism $K^\times \rightarrow F^\times$.
- Prove that $N_{K/F}(a\alpha) = a^n N_{K/F}(\alpha)$ where $a \in F$ and $n = [K : F]$.
- Let $K = F(\sqrt{D})$ be a quadratic extension of a field F whose characteristic is not 2. Show that $N_{K/F}(a + b\sqrt{D}) = a^2 - Db^2$, where $a, b \in F$.
- Let K be the splitting field of $x^3 - 2$ over \mathbb{Q} . Compute $N_{K/\mathbb{Q}}$ for $\alpha = \sqrt[3]{2}$ and for $\zeta = \zeta_3$.

Note: It is possible to define $N_{K/F}(\alpha)$ even when K/F is not Galois, see Problem #17 in Section 14.2 of our text.