

Math 500: //DRAFT ONLY// HW 8 due Friday, April 4, 2025.

Webpage: <http://dunfield.info/500>

Office hours: Wednesdays 1:30-2:30pm and Thursdays 2:00-3:00pm; additional times possible by appointment.

1. Let R be a commutative ring with 1. Prove that M and $\text{Hom}_R(R, M)$ are isomorphic as R -modules.
2. (a) Suppose M is a \mathbb{Z} -module. If $(M, +)$ has a nontrivial element m of finite order, show that M is not a free module on any subset S .
(b) Prove or disprove: the same result holds with \mathbb{Z} replaced by any field \mathbb{F} .
3. Suppose R is an integral domain. Prove that any free R -module is torsion free.
4. Let R be an integral domain and M any non-principal ideal of R such that $M \neq (0)$, considered as an R -module. Show that (i) M is torsion free (ii) $\text{rank } M = 1$, but (iii) M is not a free module. (Hint: Given $f \in M$, when is $M/(Rf)$ a torsion module? See also problem 12.1.5 in [DF].)
5. Let R be a PID with fraction field F , and note that F has the structure of an R -module (by multiplication). Show that the rank of F as an R -module is equal to 1. (Hint: think about the R -module F/R .)
6. Let R be a PID, B a torsion R -module, and p a prime element in R . Prove that if $pb = 0$ for some $b \in B$, $b \neq 0$, then $\text{Ann}(B) \subseteq (p)$. (Recall that $\text{Ann}(B) = \{r \in R \mid rB = 0\}$.)
7. Let $R = \mathbb{Z}[i]$. Classify all finitely generated R -modules M with the property that $5M = 0$.
8. Use the rational canonical form to determine all conjugacy classes in the group $G = \text{GL}_2\mathbb{F}_3$. Specifically, give a list of matrices in rational canonical form, one in each conjugacy class. Warning: $\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$ is *not* in rational canonical form, even though it's diagonal.
9. Find all similarity classes of 6×6 matrices over \mathbb{Q} with minimal polynomial $(x + 2)^2(x - 1)$. (It suffices to give all lists of invariant factors and write out some of their corresponding matrices).
10. **This problem has been removed because we won't cover this material until Friday.** Compute the Jordan canonical form (over \mathbb{C}) of
$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -2 \\ 0 & 1 & 3 \end{bmatrix}.$$