Math 500: //DRAFT ONLY// HW 5 due Friday, March 7, 2025.

Webpage: `http://dunfield.info/500`
Office hours: Wednesdays 1:30–2:30pm and Thursdays 2:00–3:00pm; additional times possible by appointment.

1. The center of a ring is $\text{Center}(R) := \{z \in R \mid rz = zr \ \ \forall\, r \in R\}$. Show that the center $\text{Center}(R)$ is a subring of R, which contains the identity of R if it has one. Show that the center of a division ring is a field.

2. Let R be a commutative ring with identity. Let $S \subseteq M_{2\times 2}(R)$ be the set of upper triangular 2×2 matrices with entries in R. Show that S is a subring of $M_{2\times 2}(R)$. Show that there is a surjective ring homomorphism $S \to R \times R$ and describe its kernel.

3. Recall the Hamilton quaternions $\mathbb{H} = \{a + bi + cj + dk \mid a, b, c, d \in \mathbb{R}\}$. For $x = a+bi+cj+dk$, define $\overline{x} := a - bi - cj - dk$.

   (a) Prove that $N(x) := x\overline{x} = a^2 + b^2 + c^2 + d^2$, and that $N(xy) = N(x)N(y)$ for $x, y \in \mathbb{H}$.

   (b) Let $\mathcal{O} \subseteq \mathbb{H}$ be the subring of integral quaternions (i.e., $a+bi+cj+dk$ such that $a, b, c, d \in \mathbb{Z}$). Prove that $\mathcal{O}^\times = \{x \in \mathcal{O} \mid N(x) = \pm 1\} \approx Q_8$. (Hint: use that $N(x) \in \mathbb{Z}$ if $x \in \mathcal{O}$.)

   (c) Determine $\text{Center}(\mathbb{H})$.

4. Prove that if $I_1 \subseteq I_2 \subseteq \cdots \subseteq I_k \subseteq \cdots$, $k \in \mathbb{Z}_{>0}$, is a chain of ideals of a ring R, then $J := \bigcup_{k=1}^{\infty} I_k$ is also an ideal.

5. Let R be a commutative ring with 1. Let I, J, P be ideals of R, with P a prime ideal. Show that if $IJ \subseteq P$ then either $I \subseteq P$ or $J \subseteq P$.

6. Let $\phi\colon R \to S$ be a ring homomorphism.

   (a) Prove that if J is an ideal of S, then $\phi^{-1}J$ is an ideal of R.

   (b) Prove that if ϕ is surjective and I an ideal of R, then $\phi(I)$ is an ideal of S. Give an example where this fails if ϕ is not surjective.

7. This problem has been removed as its solution is included in [R2].

8. Let R be a commutative ring, and let $N(R) := \{x \in R \mid x^n = 0 \text{ for some } n \in \mathbb{Z}_{>0}\}$. Show that N is an ideal.

   Note: Remember to check that N is closed under addition; for a hint, see Prob. 7.3.29 in [DF].

9. Let R be a commutative ring with 1. Show that (a) $N(R/N(R)) = 0$ and (b) that $N(R)$ is contained in the intersection of all prime ideals of R.

10. Let R be a domain, and let $N\colon R \to \mathbb{Z}$ be a function such that (i) $N(a) = 0$ iff $a = 0$, $N(1) = 1$, and $N(ab) = N(a)N(b)$ for all $a, b \in R$, and (ii) $N(a) \in \mathbb{Z}^\times$ implies $a \in R^\times$. Show that if $N(a) = \pm p$ for some prime integer p, then a is an irreducible element of R. Use this to show that $3 + 2i$ is an irreducible element of $\mathbb{Z}[i]$.

Credit: Problems 7 and 10 are from [R] and the rest from [DF].