Math 500: HW 4 due Friday, February 21, 2025.

Webpage: http://dunfield.info/500
Office hours: Wednesdays 1:30–2:30pm and Thursdays 2:00–3:00pm; additional times possible by appointment.
Textbook: Dummit and Foote, *Abstract Algebra*, 3rd edition.
Suplemental text: Charles Rezk, Lecture Notes for Math 500, posted on our course webpage.

1. Let G be the set $\mathbb{Z}^2$, with binary operation defined by

$$(x_1, y_1) \cdot (x_2, y_2) := (x_1 + x_2,\ y_1 + y_2 + x_1 x_2).$$

Show that $(G, \cdot)$ is a finitely generated abelian group, and determine its invariant factor form.

2. A subgroup $H \leq G$ is *characteristic* if $\phi(H) = H$ for every $\phi \in \mathrm{Aut}(G)$.
   (a) Prove than every characteristic subgroup is normal. Also, give an example showing that a normal subgroup need not be characteristic.
   (b) Show that if $H, K \leq G$ are characteristic subgroups with $HK = G$ and $H \cap K = \{e\}$, then $\mathrm{Aut}(G) \approx \mathrm{Aut}(H) \times \mathrm{Aut}(K)$.

3. Recall that a group H is *simple* when its only normal subgroups are $\{e\}$ and H itself.
   (a) Let A and B be non-abelian simple groups, and let $G = A \times B$. Show that the only normal subgroups of G are $\{e \times e\}$, $\{e\} \times B$, $A \times \{e\}$, and G itself. **Warning:** See the next part!
   (b) Show that the conclusion in (a) does not hold for *abelian* simple groups A and B.
   (c) Suppose now that $G_1, \ldots, G_n$ are non-abelian simple groups and let $G = G_1 \times \cdots \times G_n$. What are all the possible normal subgroups of G? You do not need to prove your answer.

4. Let G be the set $\mathbb{Z}^3$ with binary operation defined by

$$(x_1, y_1, z_1) \cdot (x_2, y_2, z_2) := (x_1 + x_2,\ y_1 + y_2,\ z_1 + z_2 + y_1 x_2).$$

Show that G is a non-abelian group. Then identify an infinite cyclic normal subgroup $H \leq G$ such that G/H is isomorphic to a product of two infinite cyclic groups. (That is, show that G is an extension of $\mathbb{Z}^2$ by $\mathbb{Z}$.) Is this extension split?

5. Let $A = \mathbb{Z}/30 \times \mathbb{Z}/45 \times \mathbb{Z}/12 \times \mathbb{Z}/18$, a finite abelian group of order 291,600.
   (a) Find the invariant factor decomposition of A as in Theorem 3 of Section 5.2 of [DF].
   (b) Find the primary decomposition of A as in Theorem 5 of Section 5.2 of [DF]. (Note: [DF] refers to this as the "elementary divisor decomposition".)
   (c) Find the number of elements of A of order 2.
   (d) Find the number of subgroups of A of index 2.

6. Suppose G is a semidirect product of $H \trianglelefteq G$ and $K \leq G$, with $\phi\colon K \to \mathrm{Aut}(H)$ defined by conjugation. Show that $C_G(H) \cap K = \ker(\phi)$ and $C_G(K) \cap H = N_G(K) \cap H$.

7. Show that there are exactly 4 distinct homomorphisms $C_2 \to \mathrm{Aut}(C_8)$. Prove that the resulting semidirect products give 4 nonisomorphic groups of order 16. Hint: Count the number of elements of order 2.

8. Show that D_n is nilpotent if and only if n is a power of 2.

9. Let $G = \mathrm{SL}_2\mathbb{F}_3$. Determine (i) the subgroups $G^{(k)}$ in the derived series of G, and (ii) the subgroups $Z_k(G)$ in the upper central series of G. In both cases, determine the quotient groups $G^{(k-1)}/G^{(k)}$ and $Z_k(G)/Z_{k-1}(G)$ up to isomorphism.

Credit: Problems 2, 4, 6, and 8 are from [DF] and Problems 1, 5, and 7 from [R].