Math 500: HW 3 due Friday, February 14, 2025.

Webpage: http://dunfield.info/500
Office hours: Wednesdays 1:30–2:30pm and Thursdays 2:00–3:00pm; additional times possible by appointment.
Textbook: Dummit and Foote, *Abstract Algebra*, 3rd edition.
Suplemental text: Charles Rezk, Lecture Notes for Math 500, posted on our course webpage.

1. Show that $|\mathrm{Aut}(D_8)| \leq 8$, by constraining the possible images of r, s under an automorphism. Then show that $\mathrm{Aut}(D_8) \approx D_8$ using that D_8 is isomorphic to a normal subgroup of D_{16}. Determine $\mathrm{Inn}(D_8)$ and $\mathrm{Out}(D_8)$.

2. Exhibit all 2 and 3 Sylow subgroups of $S_3 \times S_3$ and D_{12}.

3. Show that if $|G| = 105$, then G has a normal cyclic subgroup of index 3.

4. Show that if $|G| = 315$ such that G has a normal 3-Sylow subgroup, then its center Z_G contains a 3-Sylow subgroup. Deduce that G is abelian.

5. Let $P \in \mathrm{Syl}_p(H)$, and $H \leq K$. If $P \trianglelefteq H$ and $H \trianglelefteq K$, prove that P is normal in K. Deduce that if $P \in \mathrm{Syl}_p(G)$ and $H = N_G(P)$, then $N_G(H) = H$ (i.e., normalizers of Sylow p-subgroups are self normalizing.)

6. Let P be a normal Sylow p-subgroup of G, and $H \leq G$. Prove that $P \cap H$ is the unique Sylow p-subgroup of H.

7. Let $P \in \mathrm{Syl}_p(G)$ and assume $N \trianglelefteq G$. Use the conjugacy part of Sylow's theorem to prove that $P \cap N \in \mathrm{Syl}_p(N)$. Deduce that $PN/N \in \mathrm{Syl}_p(G/N)$.

8. Let A, B be finite groups, and p a prime number which divides the orders of both. Show that every p-Sylow subgroup of $A \times B$ is of the form $P \times Q$, with $P \in \mathrm{Syl}_p(A)$ and $Q \in \mathrm{Syl}_p(B)$. Conclude that $n_p(A \times B) = n_p(A) \times n_p(B)$. Note: See Chapter 5.1 of [DF] for background on direct products.

9. Show that every finitely generated subgroup of $\mathbb{Q}$ is cyclic.

10. Let $(P, \leq)$ be a partially ordered set. Show that $(P, \leq)$ has the ascending chain condition if and only if every non-empty subset $S \subseteq P$ has a maximal element (that is, $\exists\ m \in S$ such that for all $s \in S$, $m \leq s$ implies $m = s$).

Credit: Problems 3, 9, and 10 are from [R] and the rest from [DF].