Math 500: HW 2 due Friday, February 7, 2025.

Webpage: http://dunfield.info/500
Office hours: Wednesdays 1:30–2:30pm and Thursdays 2:00–3:00pm; additional times possible by appointment.
Textbook: Dummit and Foote, *Abstract Algebra*, 3rd edition.
Suplemental text: Charles Rezk, Lecture Notes for Math 500, posted on our course webpage.

1. Let S_3 act on the set Ω of ordered pairs (i, j) with $1 \leq i, j \leq 3$, by $\sigma((i, j)) := (\sigma(i), \sigma(j))$. Let $\phi\colon S_3 \to \mathrm{Sym}(\Omega)$ denote the corresponding homomorphism.
   (i) Find the orbits of this action.
   (ii) For each $\sigma \in S_3$, find the cycle decomposition of $\phi(\sigma) \in \mathrm{Sym}(\Omega)$.
   (iii) For each orbit $\mathcal{O}$ of S_3 acting on Ω, pick some $a \in \mathcal{O}$ and describe its stabilizer subgroup $\mathrm{Stab}(a) \subseteq S_3$.

2. Let G be a group and m a positive integer. Show that the following are equivalent.
   (a) G acts transitively on some set X of size m.
   (b) There exists a subgroup $H \leq G$ with $|G : H| = m$.

3. Let G be a group with subgroup H, and let $\lambda\colon G \to \mathrm{Sym}(G/H)$ be the left-coset action, defined by $\lambda_g(xH) := gxH$. Show that $K := \mathrm{Ker}\,\lambda = \bigcap_{x \in G} xHx^{-1}$.

4. Let G be a finite group with a subgroup H of index m, and let $K := \mathrm{Ker}\,\lambda$ as in the previous problem.
   (a) Show that $|H : K|$ divides $(m - 1)!$.
   (b) Use this to show that if p is the smallest prime which divides $|G|$, then any subgroup of index p in G is normal.

5. Suppose $p, m, a \in \mathbb{Z}_{>0}$ with p prime. In this problem, you will show $\binom{p^a m}{p^a} \equiv m \bmod p$ using the Orbit Stabilizer Theorem. Consider the left action of $G = C_{p^a} = \langle r \rangle$ on $X = \{1, 2, \ldots, p^a m\}$ defined by setting $r \cdot k$ to be $k + m \pmod{p^a m}$. (If you view the points of X as arranged evenly along a circle, then r is rotation by $2\pi/p^a$.) Now let Y be the collection of all unordered subsets of X of size p^a, so that $|Y| = \binom{p^a m}{p^a}$. Use the induced action of G on Y to prove $|Y| \equiv m \bmod p$.

6. Let G be a finite group of order $n = |G|$, and suppose there exists a *minimal non-trivial subgroup*, i.e., a subgroup $\{e\} \neq M \leq G$ such that $M \leq H$ for any $H \leq G$ with $H \neq \{e\}$. Show that G is not isomorphic to any subgroup of S_{n-1}. (Hint: Orbit Stabilizer Theorem.) Use this to show that the quaternion group of order 8 is not isomorphic to a subgroup of S_7.

7. Find all conjugacy classes and their sizes in the groups: D_8, Q_8, A_4.

8. Find all finite groups which have exactly two conjugacy classes. (The conjugacy class of the identity element counts as one of these.)

9. If G is a finite group of odd order, show that for $x \in G \setminus \{e\}$, the elements x and x^{-1} are not conjugate.

Credit: Problems 1, and 7 are from [DF], the rest from [R].