

Math 500: HW 1 due Friday, January 31, 2025.

Webpage: <http://dunfield.info/500>

Office hours: Wednesdays 1:30-2:30pm and Thursdays 2:00-3:00pm; additional times possible by appointment.

Textbook: Dummit and Foote, *Abstract Algebra*, 3rd edition.

Supplemental text: Charles Rezk, Lecture Notes for Math 500, posted on our course webpage.

1. Let G be a group. Suppose $a, b \in G$ are such that $|a| = m$, $|b| = n$, and $ab = ba$. Show that if $\gcd(m, n) = 1$, then $c := ab$ has order mn . (Here $|a|$ is the order of a .)
2. Show that if an element a in a group G has finite order m , then for any positive integer k such that $\gcd(k, m) = 1$, there exists an element $x \in G$ such that $x^k = a$.
3. Prove that if G is a group such that $a^2 = e$ for all $a \in G$, then G is abelian.
4. Consider $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}$ in $G = \text{GL}_2\mathbb{R}$. Compute the orders of elements A, B, AB, BA . Fun fact: The subgroup $\langle A, B \rangle$ of G turns out to be
$$\text{SL}_2\mathbb{Z} := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a, b, c, d \in \mathbb{Z} \text{ and } ad - bc = 1 \right\}.$$
5. Give an example of subgroups $K \leq H \leq G$ such that K is normal in H , H is normal in G , but K is not normal in G . (Hence “is normal subgroup” is not a transitive relation.)
6. Show $G = \text{SL}_2(\mathbb{Z}/3)$ has $|G| = 24$ and give examples of subgroups of G of orders 3, 8, and 6. Hint: see [DF] §2.4 #9-10.
7. Show that every non-identity element of a free group has infinite order. (Hint: need to use the description of elements of free group in terms of reduced words.)
8. Give a presentation of S_4 with 2 generators, and prove that your presentation is correct.
9. Prove that $\langle a, b \mid a^2b^{-2}, aba^{-1}b \rangle$ is a presentation of the quaternion group of order 8.
10. Show that $\langle a, b \mid aba^{-1}b^{-2}, bab^{-1}a^{-2} \rangle \cong \{e\}$.

Credit: Problems 3 and 7-9 are from [DF], the rest from [R].