

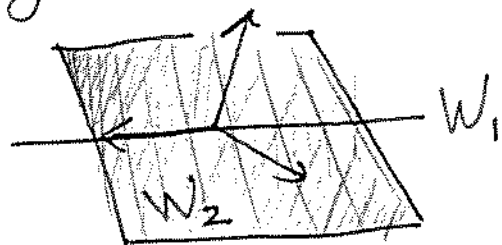
Lecture 8: Linear dependence and independence ①

[§ 1.5 of FIS]

Suppose W is a subspace of a vector space V .

How do we answer questions such as:

- ① Given w_1, w_2, \dots, w_k in W , is $\text{span}\{w_1, \dots, w_k\}$ all of W ?
- ② What is smallest number of w_i needed $\text{span } W$? How can we find such vectors?
- ③ Given v in V , is v in W ?
- ④ In \mathbb{R}^3 , we've seen 4 kinds of subspaces:
 - i) $\{0\}$
 - ii) \mathbb{R}^3
 - iii) lines through 0
 - iv) planes through 0 .



How do we distinguish these mathematically and generalize to \mathbb{R}^n ?

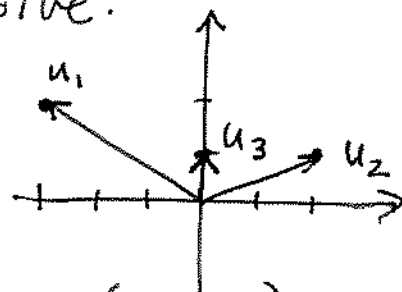
Vectors u_1, \dots, u_k in V are linearly dependent ②
if there are scalars a_1, a_2, \dots, a_k , not all 0,
such that $a_1 u_1 + \dots + a_k u_k = 0$.

Ex: $V = \mathbb{R}^2$ $u_1 = (-3, 2)$ $u_2 = (2, 1)$ $u_3 = (0, 1)$

Are these dependent? Trying to solve:

$$a_1 u_1 + a_2 u_2 + a_3 u_3 = 0$$

$$(-3a_1 + 2a_2, 2a_1 + a_2 + a_3) = (0, 0)$$



Gives 2 equations

$$\begin{aligned} -3a_1 + 2a_2 &= 0 \\ 2a_1 + a_2 + a_3 &= 0 \end{aligned}$$

So we're trying to find the null space,

of $A = \begin{pmatrix} -3 & 2 & 0 \\ 2 & 1 & 1 \end{pmatrix}$. Turns out, this

is $\{(2t, 3t, -7t) \mid t \in \mathbb{R}\}$. So these

are dependent.

A linear dependency means there is a redundancy amongst the vectors in terms of the span: $2u_1 + 3u_2 - 7u_3 = 0$ ③

$$\Rightarrow u_1 = -\frac{3}{2}u_2 + \frac{7}{2}u_3$$

and so $\text{span}(u_1, u_2, u_3) = \text{span}(u_2, u_3)$

In contrast $a_2u_2 + a_3u_3 = 0$ has only the trivial solution [easy from earlier eqn's] and so u_1, u_2 are called linearly independent.

More generally, we say $S \subseteq V$ is linearly dependent if there

are distinct vectors u_1, \dots, u_k in S which are linearly dependent; otherwise call S linearly independent.

Notes: ① The set $\{0\}$ is linearly dependent, as $a \cdot 0 = 0$ for all scalars a .

② If $u \neq 0$, then $\{u\}$ is linearly independent. as $au = 0$ for $a \neq 0$ gives $u = \frac{1}{a}(au) = \frac{1}{a}0 = 0$.

③ The empty set \emptyset is linearly independent.

Aside: By convention $\text{span}(\emptyset) = \{0\}$

Thm Suppose u_1, \dots, u_k are nonzero vectors in V , and consider the subsp $W = \text{span}(\{u_i\})$.


Then there exists a linearly independent subset $u_{i_1}, \dots, u_{i_\ell}$ of the u_i where $\text{span}(\{u_{i_1}, \dots, u_{i_\ell}\})$ is all of W .

Pf: Suppose the $\{u_i\}$ are linearly dependent.

Then we can express one of the u_i as a linear combination of the others. Reindexing the u_i if necessary, we can arrange that

$$u_k = a_1 u_1 + \dots + a_{k-1} u_{k-1}$$

In particular, $W = \text{span}(\{u_1, \dots, u_{k-1}\})$

Repeating this argument, we eventually arrive at $\underbrace{u_1, \dots, u_\ell}_{\text{relabelled many times!}}$ which are linearly independent and which span W . 

Note: Always end up with at least one u_{i_1} .

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Ex: $V = \mathbb{R}^3$ $W = \text{span}(\{u_1, u_2, u_3\})$

$u_1 = (-1, 1, 2)$ Q: What is W geometrically?

$u_2 = (1, 2, 1)$ [Should be plane or \mathbb{R}^3]

$u_3 = (5, 1, -4)$

Are these linearly dependent? The condition

$$a_1 u_1 + a_2 u_2 + a_3 u_3 = 0$$

gives rise to the eqns

$$-a_1 + a_2 + 5a_3 = 0$$

$$a_1 + 2a_2 + a_3 = 0$$

$$2a_1 + a_2 - 4a_3 = 0$$

Equivalently, we need the nullspace of

$$A = \begin{pmatrix} -1 & 1 & 5 \\ 1 & 2 & 1 \\ 2 & 1 & -4 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ u_1 & u_2 & u_3 \\ 1 & 1 & 1 \end{pmatrix}$$

useful shortcut!

which row reduces to:

and so the sol'ns $\begin{pmatrix} 1 & 0 & -3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix}$

are

$$\{3t, -2t, t \mid t \in \mathbb{R}\}$$

Taking $t=1$, get $3u_1 - 2u_2 + u_3 = 0$

$$\Rightarrow u_3 = -3u_1 + 2u_2.$$

So $W = \text{span}(\{u_1, u_2\})$ is a plane.