Lecture 7: Solution spaces to linear systems

[§TSS of Breezer]

Previously...

Thm: If M and N are row equivalent matrices then the linear systems $\mathcal{LS}(M)$ and $\mathcal{LS}(N)$ have the same set of solutions.

Thm: Any matrix is row equivalent to one in reduced row echelon form (RREF) where

1. rows of zeros at bottom
2. other rows have leading 1s.
3. a column containing a leading 1 is otherwise 0.
4. leading 1s move down and to the right.

Upshot: To understand sol'ns to linear systems it is enough to handle those whose augmented matrix is in RREF.

Ex:

$$\begin{pmatrix} \boxed{1} & 2 & 0 & 1 & 0 & 2 \\ 0 & 0 & \boxed{1} & 3 & 0 & 3 \\ 0 & 0 & 0 & 0 & \boxed{1} & 4 \end{pmatrix} \longleftrightarrow \begin{aligned} x_1 + 2x_2 \quad + x_4 \quad &= 2 \\ x_3 + 3x_4 \quad &= 3 \\ x_5 &= 4 \end{aligned}$$

Rewrite as

$$\left.\begin{aligned} x_1 &= 2 - 2x_2 - x_4 \\ x_3 &= 3 - 3x_4 \\ x_5 &= 4 \end{aligned}\right\}$$

Key: No x_1, x_3 or x_5 on this side because of rule 3 of RREF.

that is, solve for the variables corresp. to the leading 1s. Will view x_2 and x_4 as "free variables".

Thus the solution set to these eqns is: "where" "in"

$$\{(2-2s-t,\ s,\ 3-3t,\ t,\ 4) \mid s,t \in \mathbb{R}\}$$

Ex: $\begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \end{pmatrix} \longleftrightarrow \begin{aligned} x_1 &= 2 \\ x_2 &= 3 \end{aligned}$ Sol'n set $\{(2,3)\}$

Ex: $\begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 3/2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \longleftrightarrow \begin{aligned} x_1 \quad - x_3 &= 0 \\ x_2 + \tfrac{3}{2}x_2 &= 0 \\ 0 &= 1 \end{aligned}$

No solutions: Solution set is $\emptyset$.

empty set.

A linear system is <u>consistent</u> when it has at least one sol'n; otherwise it is <u>inconsistent</u>.

A <u>pivot column</u> of a matrix in RREF is one containing a leading 1.

<u>Thm</u>: If M is in RREF, then $\mathcal{LS}(M)$ is inconsistent if and only if the rightmost column is a pivot column.

<u>Pf</u>: If the rightmost column is pivot, we have
$$\begin{pmatrix} & & & 0 \\ & & & \vdots \\ & & & 0 \\ 0 & \cdots & 0 & 1 \\ & & & 0 \\ & & & \vdots \\ & & & 0 \end{pmatrix}$$
and so the lin sys has the eqn $0 = 1$ which has no solutions.

Suppose instead the rightmost column is not pivot. Let $d_1, d_2, \ldots, d_k$ be the indices of the pivot columns, and $b_1, \ldots, b_m$ be the entries of the rightmost column.

Ex:

$$\begin{pmatrix} 0 & \boxed{1} & 5 & 0 & 0 & 3 \\ 0 & 0 & 0 & \boxed{1} & 0 & 0 \\ 0 & 0 & 0 & 0 & \boxed{1} & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$d_1 = 2$ $\quad$ $b_1 = 3$

$d_2 = 4$ $\quad$ $b_2 = 0$

$d_3 = 5$ $\quad$ $b_3 = 2$

$b_4 = 0$

Claim: $x_{d_1} = b_1,\ x_{d_2} = b_2,\ \ldots,\ x_{d_k} = b_k$

and all other $x_j = 0$ is a solution to $\mathcal{LS}(M)$.

Reason: Each eqn has the form

$$x_{d_i} + (\text{terms not involving any } x_{d_\ell}) = b_i$$

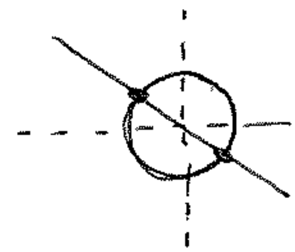
So when the rightmost column is not pivot

$\mathcal{LS}(M)$ has at least one solution,

as desired. $\blacksquare$

Thm: Any linear system has either no sol'ns, exactly one solution, or infinitely many sol'ns.

<u>Contrast</u>: $\begin{array}{l} x^2+y^2=2 \\ x+y=0 \end{array}$ has exactly <u>two</u> solutions, namely $(1,-1)$ and $(-1,1)$.



<u>Proof Idea</u>: Can assume that our $m\times(n+1)$ matrix M is in RREF with no zero rows. If column $n+1$ is pivot, then the system has no solutions and we're done. As every row has a leading 1 and the last col is non-pivot, we have $m \le n$.

Two cases:

<u>$m=n$</u>: In this case, $M = \begin{pmatrix} 1 & 0 & & & b_1 \\ 0 & 1 & 0 & 0 & \vdots \\ 0 & 0 & 1 & & \vdots \\ & & & \ddots & \\ 0 & 0 & 0 & 1 & b_m \end{pmatrix}$

which corresponds to $\begin{array}{l} x_1 = b_1 \\ \vdots \\ x_n = b_n \end{array} \implies$ unique sol'n.

<u>$m<n$</u>: In this case, there is at least one non-pivot column. If we assign any real values to the non-pivot variables,

we can always solve for the pivot variables as we did before. Thus we get infinitely many solutions in this case. $\square$

<u>Note</u>: In the last case, we need $n-m$ "parameters" to write down all possible solutions to the linear system.

<u>Next time</u>: Begin making precise this notion of "number of parameters" in the context of subspaces of a vector space.