Lecture 5: Using matrices to solve linear systems

Reference: [§RREF of Breezer]

Last time: System of linear equations:

Variables: $x_1, x_2, \dots, x_n$

Equations:
$$\begin{aligned} a_{11}x_1 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + \cdots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{m1}x_1 + \cdots + a_{mn}x_n &= b_m \end{aligned}$$

where a_{ij} and b_i are in $\mathbb{R}$.

To a linear system, we associate a coefficient matrix $A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \\ a_{m1} & \cdots & a_{mn} \end{pmatrix}$,

a vector of constants $b = \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix}$ ← column vector

and an augmented matrix

$$\begin{pmatrix} a_{11} & \cdots & a_{1n} & b_1 \\ \vdots & & & \vdots \\ a_{m1} & \cdots & a_{mn} & b_m \end{pmatrix}$$

Ex: $$\begin{aligned} x_1 + 2x_2 + 2x_3 &= 4 \\ x_1 + 3x_2 + 3x_3 &= 5 \\ 2x_1 + 6x_2 + 5x_3 &= 6 \end{aligned}$$

This system has $A = \begin{pmatrix} 1 & 2 & 2 \\ 1 & 3 & 3 \\ 2 & 6 & 5 \end{pmatrix}$ and $b = \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix}$.

The augmented matrix $(A|b)$ is $\begin{pmatrix} 1 & 2 & 2 & 4 \\ 1 & 3 & 3 & 5 \\ 2 & 6 & 5 & 6 \end{pmatrix}$.

Decoding:

Given an $m \times n$ matrix A and an m entry column vector b, we associate a linear system $\mathcal{LS}(A, b)$. Also works for augmented matrices.

Ex: $\mathcal{LS}\left(\begin{pmatrix} 1 & 3 & 1 \\ 2 & 1 & 0 \end{pmatrix}\right) = \begin{cases} x_1 + 3x_2 = 1 \\ 2x_1 + x_2 = 0 \end{cases}$

Row operations: These take a matrix M and give a new matrix of the same size.

1. Swap two rows of M

2. Multiply all entries in a row by some <u>nonzero</u> c in $\mathbb{R}$

3. Add a multiple of one row to another.

[Hint at use for solving lin. systems.]

<u>Ex</u>: $M = \begin{pmatrix} 1 & 2 & 2 & 4 \\ 1 & 3 & 3 & 5 \\ 2 & 6 & 5 & 6 \end{pmatrix} \xrightarrow{R_2 \leftrightarrow R_3} \begin{pmatrix} 1 & 2 & 2 & 4 \\ 2 & 6 & 5 & 6 \\ 1 & 3 & 3 & 5 \end{pmatrix}$

$\xrightarrow{-3R_3} \begin{pmatrix} 1 & 2 & 2 & 4 \\ 2 & 6 & 5 & 6 \\ -3 & -9 & -9 & -15 \end{pmatrix} \xrightarrow{3R_1 + R_3} \begin{pmatrix} 1 & 2 & 2 & 4 \\ 2 & 6 & 5 & 6 \\ 0 & -3 & -3 & -3 \end{pmatrix}$

$\xrightarrow[-\frac{1}{3}R_3]{} \begin{pmatrix} 1 & 2 & 2 & 4 \\ 2 & 6 & 5 & 6 \\ 0 & 1 & 1 & 1 \end{pmatrix}$

<u>Def</u>: A matrix M is <u>row equivalent</u> to N if there is a sequence of row operations that turns M into N.

<u>Note</u>: On HW you'll show this is symmetric in M and N.

Thm: If M is row equivalent to N, then the linear systems $\mathcal{LS}(M)$ and $\mathcal{LS}(N)$ have the same solution sets.

Pf: Just need to check that a single row op does not change the solution set of the corresp. linear system. Row op 1 is fine. [since it corresponds to reordering two equations] Row op 2 just scales the assoc. eqn. For row op 3, it is enough to consider the case of adding rows without any scaling. [The general case follows by using row op 2]

On the equation side, have

$$E_1 \quad \begin{aligned} a_1x_1 + \cdots + a_nx_n &= b \\ a_1'x_1 + \cdots + a_n'x_n &= b' \end{aligned} \quad \rightsquigarrow$$

$$E_2 \quad \begin{aligned} a_1x_1 + \cdots \quad a_n \; x_n &= b \\ (a_1'+a_1)x_1 + \cdots + (a_n'+a_n)x_n &= b+b' \end{aligned}$$

If numbers $(x_1, \ldots, x_n)$ satisfy E_1 then they also sat $\tilde{E}_2$. Conversely if they sat E_2 they also sat E_1 since we can subtract the first eqn from the second equation of $\tilde{E}_2$.

So all three kinds of row ops don't change the sol set of the corresp linear sys. ∎

Ex. (Continuation)

$$M \xrightarrow[\text{ops}]{\text{row}} \begin{pmatrix} 1 & 2 & 2 & 4 \\ 2 & 6 & 5 & 6 \\ 0 & 1 & 1 & 1 \end{pmatrix} \xrightarrow{-2R_1+R_2} \begin{pmatrix} 1 & 2 & 2 & 4 \\ 0 & 2 & 1 & -2 \\ 0 & 1 & 1 & 1 \end{pmatrix}$$

$$\xrightarrow{-R_2+R_1} \begin{pmatrix} 1 & 0 & 1 & 6 \\ 0 & 2 & 1 & -2 \\ 0 & 1 & 1 & 1 \end{pmatrix} \xrightarrow{-2R_3+R_2} \begin{pmatrix} 1 & 0 & 1 & 6 \\ 0 & 0 & -1 & -4 \\ 0 & 1 & 1 & 1 \end{pmatrix}$$

$$\xrightarrow[R_2+R_3]{R_2+R_1} \begin{pmatrix} 1 & 0 & 0 & 2 \\ 0 & 0 & -1 & -4 \\ 0 & 1 & 0 & -3 \end{pmatrix} \xrightarrow{-R_2} \begin{pmatrix} 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 4 \\ 0 & 1 & 0 & -3 \end{pmatrix}$$

$$\xrightarrow{LS} \begin{aligned} x_1 &= 2 \\ x_3 &= 4 \\ x_2 &= -3 \end{aligned}$$

So $(x_1, x_2, x_3) = (2, -3, 4)$

is the unique soln to our original system:

$$\begin{aligned} x_1 + 2x_2 + 2x_3 &= 4 \\ x_1 + 3x_2 + 3x_3 &= 5 \\ 2x_1 + 6x_2 + 5x_3 &= 6. \end{aligned}$$

as you can (and should!) double check.

__Goal__: Develop systematic procedure for solving such systems, by finding matrices like these:

__Reduced Row-Echelon Form__: A matrix where:

1. All zero rows are at the bottom.
2. The leftmost nonzero entry of every row is 1, and is called a __leading 1__.
3. A leading 1 is the only nonzero entry in its column.
4. Suppose entries (i,j) and (s,t) are leading 1's. If $s > i$ then $t > j$.