# Lecture 33: Gram-Schmidt and friends §6.2 (1)

Last time: $V$ an inner product space.

- $S \subseteq V$ is orthogonal if $\langle x, y \rangle = 0$ for all distinct $x, y \in S$.
- If additionally all $x$ in $S$ are unit, then $S$ is called orthonormal.

Thm: Suppose $S = \{v_1, v_2, \ldots, v_k\} \subseteq V$ is orthonormal. If $y \in \text{span}(S)$ then $y = \sum \langle y, v_i \rangle v_i$. Moreover, $S$ is linearly independent.

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Thm: Suppose $V$ is a finite dim'l inner product space. Then $V$ has an orthonormal basis.

[Prove today via the algorithm that builds such a basis...]

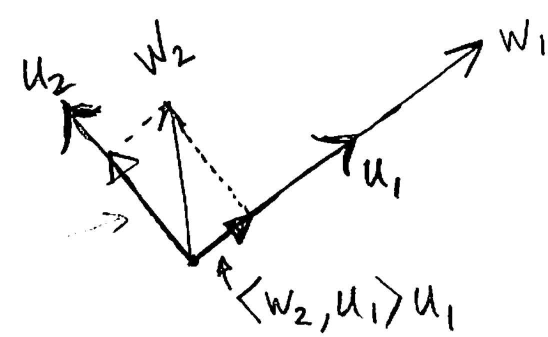
Gram-Schmidt Process: $\{w_1, \ldots, w_n\}$ linearly indep.

$$u_1 = \text{unit}(w_1) = \frac{w_1}{\|w_1\|}$$

$$u_2 = \text{unit}(w_2 - \langle w_2, u_1 \rangle u_1)$$

$$\vdots$$

$$u_k = \text{unit}\left(w_k - \sum_{i=1}^{k-1} \langle w_k, u_i \rangle u_i\right)$$



<u>Claim</u>: $\{u_1, \dots, u_n\}$ is orthonormal with the same span as $\{w_1, \dots, w_n\}$

<u>Note</u>: By taking $\{w_1, \dots, w_n\}$ to be a basis of $V$ then proves the theorem.

<u>Pf of Claim</u>: Induct on $n$.

<u>$n=1$</u>: As $\mathrm{span}(\{u_1\}) = \mathrm{span}(\{w_1\})$ and any single unit vector is orthonormal the claim holds.

<u>$n=2$</u>: Set $v_2 = w_2 - \langle w_2, u_1 \rangle u_1$ so that $u_2 = \mathrm{unit}(v_2) = \dfrac{v_2}{\|v_2\|}$. Now $\{u_1, u_2\}$ is orthonormal as

$$\begin{aligned}\langle v_2, u_1 \rangle &= \langle w_2, u_1 \rangle - \langle \langle w_2, u_1 \rangle u_1, u_1 \rangle \\ &= \langle w_2, u_1 \rangle - \langle w_2, u_1 \rangle \underbrace{\langle u_1, u_1 \rangle}_{=1} = 0.\end{aligned}$$

As $\{u_1, u_2\}$ is linearly independent, have $\dim(\mathrm{span}\{u_1, u_2\}) = 2$. As $\mathrm{span}(\{u_1, u_2\}) \subseteq \mathrm{span}(\{w_1, w_2\})$ this forces $\mathrm{span}(\{u_1, u_2\}) = \mathrm{span}(\{w_1, w_2\})$ as needed.

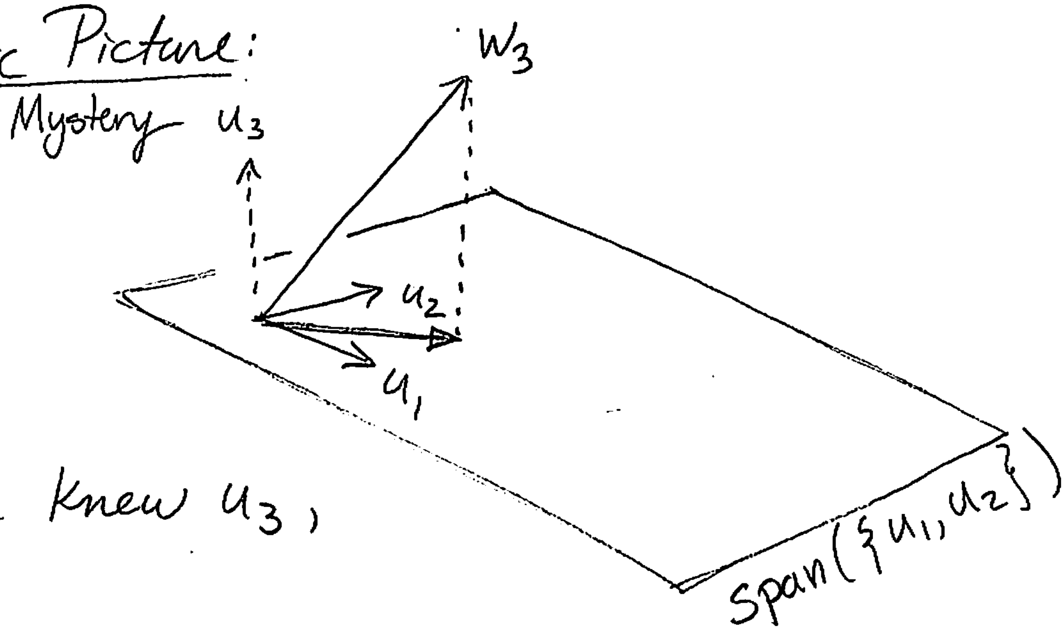
The general inductive step is essentially the same as for $n=2$:

- Check that $u_n$ is orthogonal to $\{u_1, \dots, u_{n-1}\}$.
- Note $u_n \in \operatorname{span}(\{u_1, \dots, u_{n-1}, w_n\})$

$$= \operatorname{span}(\{w_1, \dots, w_n\}) \text{ by induction}$$

and as $\{u_1, \dots, u_n\}$ is linearly indep we must have $\operatorname{span}(\{u_1, \dots, u_n\}) = \operatorname{span}(\{w_1, \dots, w_n\})$ for dimension reasons. $\blacksquare$

<u>Geometric Picture:</u>




If we knew $u_3$, then

$$W_3 = \underbrace{\langle W_3, u_1\rangle u_1 + \langle w_3, u_2\rangle u_2}_{\text{Can calculate without knowing what } u_3 \text{ is.}} + \langle w_3, u_3\rangle u_3$$

Can calculate <u>without</u> knowing what $u_3$ is.

Now solve for $u_3$!

Ex: $V = P_2(\mathbb{R})$ $\quad \langle f, g \rangle = \int_{-1}^{1} f(x)g(x)\,dx$

$$\{1, x, x^2\} \xrightarrow[\text{Schmidt}]{\text{Gram}} \left\{\frac{1}{\sqrt{2}}, \sqrt{\frac{3}{2}}\,x, \sqrt{\frac{5}{8}}\,(3x^2-1)\right\}$$

$$\langle 1, 1 \rangle = \int_{-1}^{1} 1\cdot 1\,dx = 2 \Rightarrow \|1\| = \sqrt{2}$$

These are called Legendre polynomials, the next few are $\sqrt{\frac{7}{8}}(5x^3-3x)$, $\sqrt{\frac{9}{128}}(35x^4-30x^2+3)$, ...

[First appeared in study of series expansions of gravitational potential functions in spherical coor.]

Def: Suppose $S \subseteq V$ is orthonormal. For $x \in V$, the scalars $\langle x, u \rangle$ with $u \in S$ are called the Fourier coefficients of $x$ relative to $S$.

Reason for terminology: $V = \overbrace{\mathcal{C}([-1,1])}^{\text{continuous, real valued fns.}}$

and $\langle f(x), g(x) \rangle = \int_{-1}^{1} f(x)g(x)\,dt$

Then $S = \left\{\frac{1}{\sqrt{2}}\right\} \cup \{\sin(\pi n x)\}_{n=1}^{\infty} \cup \{\cos(\pi n x)\}_{n=1}^{\infty}$

is an orthonormal subset of V. Consider $h(x) = |x|$ in V. The Fourier coefficients are

$$\langle h, \tfrac{1}{\sqrt{2}} \rangle = \int_{-1}^{1} |x| \frac{1}{\sqrt{2}}\, dx = \sqrt{2} \int_0^1 x\, dx = \frac{1}{\sqrt{2}}$$

$$\langle h, \sin \pi n x \rangle = 0 \quad \text{for symmetry reasons}$$

$$\langle h, \cos \pi n x \rangle = \int_{-1}^{1} |x| \cos \pi n x\, dx = 2 \int_0^1 x \cos(\pi n x)\, dx$$

$$= \begin{cases} 0 & n \text{ even} \\ -\dfrac{4}{n^2\pi^2} & n \text{ odd} \end{cases}$$

<u>Note</u>: S is <u>not</u> a basis of V and infinitely many of these Fourier coefficients are non-zero. However, if we allow infinite sums, get

$$|x| = \frac{1}{2} + \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} -\frac{4}{n^2\pi^2} \cos(\pi n x)$$

for all $x \in [-1, 1]$. Fun corollary: $\dfrac{\pi^2}{8} = \displaystyle\sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{1}{n^2}$