

Lecture 31 Inner product spaces. (§6.1) ①

[Original motivation was the geometry/algebra of vectors in \mathbb{R}^2 and \mathbb{R}^3 . One thing that has been missing: the dot product. Let's fix that...]

Def: Suppose V is a vector space over \mathbb{R} . An inner product is a function from pairs of vectors to \mathbb{R} , denoted $\langle x, y \rangle$, such that ~~for~~ for all $x, y, z \in V$ and $c \in \mathbb{R}$ one has:

$$a) \langle x + z, y \rangle = \langle x, y \rangle + \langle z, y \rangle$$

$$b) \langle cx, y \rangle = c \langle x, y \rangle$$

$$c) \langle x, y \rangle = \langle y, x \rangle$$

$$d) \langle x, x \rangle > 0 \text{ if } x \neq 0.$$

Ex: $V = \mathbb{R}^n$, $\langle x, y \rangle = x \cdot y = \sum_{i=1}^n x_i y_i$

Ex: $V = \mathbb{R}^2$ $\langle x, y \rangle = 3x_1 y_1 + 2x_2 y_2$

Ex: $V = \mathbb{R}^2$ $\langle x, y \rangle = 2x_1 y_1 + x_1 y_2 + x_2 y_1 + x_2 y_2$

Non Ex: $V = \mathbb{R}^2$ with any of

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$$\langle x, y \rangle = x_1^2 y_1 + x_2 y_2$$

$$\langle x, y \rangle = x_1 y_1 + x_1 y_2 + x_2 y_2$$

$$\langle x, y \rangle = x_1 y_1 - x_2 y_2$$

Ex: $V = \mathcal{C}([0, 1]) = \{f: [0, 1] \rightarrow \mathbb{R} \mid f \text{ continuous}\}$

$$\langle f, g \rangle = \int_0^1 f(t)g(t) dt$$

For (d), have $\langle f, f \rangle = \int_0^1 (f(t))^2 dt$, and so if this = 0 we must have $f^2 = 0$ (as $f^2 \geq 0$) and so $f = 0$.

Def: Suppose V is a vector space over \mathbb{C} . An inner product on V is a function from pairs of vectors to \mathbb{C} satisfying the same rules with one change:

$$c') \quad \overline{\langle x, y \rangle} = \langle y, x \rangle$$

Here "—" denotes complex conjugation: $\overline{a+bi} = a-bi$

Non Ex: $V = \mathbb{C}^2$ $\langle x, y \rangle = x_1 y_1 + x_2 y_2$

Problem: $\langle (i, 0), (i, 0) \rangle = -1$

Ex: $V = \mathbb{C}^2$ $\langle x, y \rangle = x_1 \bar{y}_1 + x_2 \bar{y}_2$

Now $\langle (i, 0), (i, 0) \rangle = i \cdot \bar{i} + 0 \cdot \bar{0} = +1 > 0$.

Def: The conjugate transpose or adjoint of $A \in M_{n \times n}(\mathbb{C})$ is $A^* = \overline{(A^t)}$. That is $A^*_{ij} = \overline{A_{ji}}$

Ex: $A = \begin{pmatrix} 2+i & 3 \\ -i & 4 \end{pmatrix}$ $A^* = \begin{pmatrix} 2-i & i \\ 3 & 4 \end{pmatrix}$

For $A \in M_{n \times n}(\mathbb{R})$, A^* is just A^t . As with transpose, have $(A+B)^* = A^* + B^*$ and $(AB)^* = B^* A^*$.

The Frobenius inner product on $M_{n \times n}(\mathbb{R})$ or $M_{n \times n}(\mathbb{C})$ is defined by $\langle A, B \rangle = \text{tr}(B^* A)$.

Basic Properties: V an inner product space.

For $x, y, z \in V$ and any scalar c :

$$\langle x, y+z \rangle = \langle x, y \rangle + \langle x, z \rangle$$

$$\langle x, cy \rangle = \bar{c} \langle x, y \rangle$$

$$\langle x, 0 \rangle = \langle 0, x \rangle = 0$$

$$\langle x, x \rangle = 0 \iff x = 0$$

↖ A pair (V, \langle, \rangle)

Def: Let V be an inner product space. The norm or length of $x \in V$ is $\|x\| = \sqrt{\langle x, x \rangle} \in \mathbb{R}_{\geq 0}$.

Ex: $(\mathbb{R}^n, \text{dot product})$ Then $\|x\| = \sqrt{\langle x, x \rangle} = \left(\sum_{i=1}^n x_i^2 \right)^{1/2}$.

Thm: Suppose V is an inner product space.

For all $x, y \in V$ and scalars c one has

a) $\|cx\| = |c| \|x\|$

b) $\|x\| = 0 \iff x = 0$

$$c) |\langle x, y \rangle| \leq \|x\| \cdot \|y\| \quad (\text{Cauchy-Schwarz}) \quad (5)$$

$$d) \|x+y\| \leq \|x\| + \|y\| \quad (\Delta\text{-inequality})$$

Proof: Parts (a) and (b) are easy; will do (c) and (d) in the case where the scalars are \mathbb{R} .

c) If $y=0$, we get $0 \leq 0$ as need; so assume $y \neq 0$.

Also if scale x or y by $c \in \mathbb{R}$, then both sides change by $|c|$ as per part (a). So may assume that

$\|y\| = 1$. Now

$$\begin{aligned} 0 \leq \|x - \langle x, y \rangle y\|^2 &= \langle x - \langle x, y \rangle y, x - \langle x, y \rangle y \rangle \\ &= \langle x, x \rangle - 2\langle x, y \rangle^2 + \langle x, y \rangle^2 \langle y, y \rangle \\ &= \|x\|^2 - \langle x, y \rangle^2 \end{aligned}$$

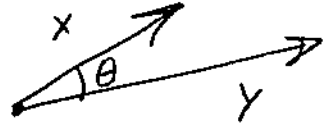
and so $|\langle x, y \rangle| \leq \|x\|$ as needed.

$$\begin{aligned} d) \|x+y\|^2 &= \langle x+y, x+y \rangle = \langle x, x \rangle + 2\langle x, y \rangle + \langle y, y \rangle \\ &\leq \langle x, x \rangle + 2|\langle x, y \rangle| + \langle y, y \rangle \\ &\leq \|x\|^2 + 2\|x\|\|y\| + \|y\|^2 \\ &= (\|x\| + \|y\|)^2 \end{aligned}$$



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Note: Motivation for (c) is that in \mathbb{R}^3 one has $x \cdot y = \|x\| \|y\| \cos \theta$



Def: Suppose V is an inner product space.

Vectors $x, y \in V$ are orthogonal/perpendicular if $\langle x, y \rangle = 0$. A subset $S \subseteq V$ is orthogonal if every pair of distinct vectors in it are orthogonal.

A vector $x \in V$ is unit if $\|x\| = 1$. Finally, a subset $S \subseteq V$ ~~is~~ which is orthogonal and consists of unit vectors is called orthonormal.

Ex: The standard basis $\{e_i\}$ is an orthonormal subset of \mathbb{R}^n with $\langle, \rangle = \text{dot product}$.