

Lecture 26: Finding eigenvectors [§5.1 and §5.2] ①

Last time: An eigenvector for a linear operator T on V is a $v \in V$ where $T(v) = \lambda v$ for some scalar λ .
nonzero \uparrow \uparrow eigenvalue assoc to v .

An eigenvector for $A \in M_{n \times n}(\mathbb{R})$ is a $v \in \mathbb{R}^n$ with $Av = \lambda v$. [Equivalently, it's an eigenvector for L_A .]
nonzero \uparrow

Today will explain how to find all eigenvectors for A . Recall:

Thm: Suppose $A \in M_{n \times n}(\mathbb{R})$. Then $\lambda \in \mathbb{R}$ is an eigenvalue for A if and only if $\det(A - \lambda I_n) = 0$.

Def: The characteristic polynomial of $A \in M_{n \times n}(\mathbb{R})$ is $f(t) = \det(A - tI_n)$.

Ex: $A = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 2 & 0 \\ -2 & 0 & 3 \end{pmatrix} \quad A - \begin{pmatrix} t & 0 & 0 \\ 0 & t & 0 \\ 0 & 0 & t \end{pmatrix} = \begin{pmatrix} -t & 0 & 1 \\ 0 & 2-t & 0 \\ -2 & 0 & 3-t \end{pmatrix}$

$$\begin{aligned} \det(A - tI) &= (2-t)(-t(3-t) + 2) \\ &= (2-t)(t^2 - 3t + 2) = -t^3 + 5t^2 - 8t + 4 \end{aligned}$$

(2)

Notes: Suppose $f(t)$ is the characteristic poly of $A \in M_{n \times n}(\mathbb{R})$.

1) For any $\lambda \in \mathbb{R}$, have $f(\lambda) = \det(A - \lambda I_n)$.

Thus

Thm: The roots of $f(t)$ are exactly the eigenvalues of A .

Ex: A as before, have char poly

$$\begin{aligned} f(t) &= (2-t)(t^2-3t+2) \\ &= -(t-2)((t-2)(t-1)) \\ &= -(t-1)(t-2)^2 \end{aligned}$$

So the eigenvalues of A are 1 and 2.

2) $f(t)$ has degree n and leading coefficient $(-1)^n$.

Cor: An $n \times n$ matrix has at most n distinct eigenvalues

(3)

Lemma (Last time) $v \in \mathbb{R}^n$ is an eigenvector

for $A \in M_{n \times n}(\mathbb{R}) \iff v \in \mathcal{N}(A - \lambda I_n)$ and $v \neq 0$.

Ex: Find all eigenvectors of $A = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 2 & 0 \\ -2 & 0 & 3 \end{pmatrix}$

with eigenvalue 1. Set $B = A - 1 \cdot I_3 = \begin{pmatrix} -1 & 0 & 1 \\ 0 & 1 & 0 \\ -2 & 0 & 2 \end{pmatrix}$

As $B \xrightarrow{-2R_1 + R_3} \begin{pmatrix} -1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, have one free var and

$$\begin{aligned} \mathcal{N}(B) &= \{(t, 0, t) \mid t \in \mathbb{R}\} \\ &= \text{span}\{(1, 0, 1)\} \end{aligned}$$

So the eigenvectors of A with eigenvalue 1 are $(t, 0, t)$ with $t \neq 0$.

Check: $\begin{pmatrix} 0 & 0 & 1 \\ 0 & 2 & 0 \\ -2 & 0 & 3 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \checkmark$

Def: If λ is an eigenvalue of $A \in M_{n \times n}(\mathbb{R})$,

define $E_\lambda = \mathcal{N}(A - \lambda I_n)$, which is called

the eigenspace of A corresponding to λ .

Note: $E_\lambda = \left\{ \begin{array}{l} \text{all eigenvectors of } A \\ \text{with eigenvalue } \lambda \end{array} \right\} \cup \{0\}$

and E_λ is a subspace of \mathbb{R}^n .

Ex: Compute E_2 for $A = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 2 & 0 \\ -2 & 0 & 3 \end{pmatrix}$.

Set $B = A - 2I_3 = \begin{pmatrix} -2 & 0 & 1 \\ 0 & 0 & 0 \\ -2 & 0 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} & s & t \\ 1 & 0 & -1/2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

$E_2 = \mathcal{N}(B) = \left\{ (t/2, s, t) \mid s, t \in \mathbb{R} \right\}$

Q: Is A diagonalizable? That is, can we find a basis β of \mathbb{R}^3 so that $[L_A]_\beta$ is a diagonal matrix?

A. Yes is equivalent to having a basis of \mathbb{R}^3 consisting of eigenvectors.

$v_1 = (1, 0, 1) \in E_1$

$\left. \begin{array}{l} v_2 = (0, 1, 0) \\ v_3 = (1, 0, 2) \end{array} \right\} \in E_2$

⑤

Then $\beta = \{v_1, v_2, v_3\}$ are a basis as if

$$Q = \begin{pmatrix} 1 & 1 & 1 \\ v_1 & v_2 & v_3 \\ 1 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 2 \end{pmatrix} \text{ then } \det(Q) = 1 \neq 0.$$

So

$$\begin{aligned} [L_A]_{\beta} &= [I_{\mathbb{R}^3}]_{\text{std}}^{\beta} [L_A]_{\text{std}} [I_{\mathbb{R}^3}]_{\beta}^{\text{std}} \\ &= Q^{-1} A Q = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} \end{aligned}$$

and hence A is diagonalizable.

Note: $A \in M_{n \times n}(\mathbb{R})$ is diagonalizable if and only if A is similar to a diagonal matrix.

Today focused on matrices, but can also consider

Def: Suppose T is a linear operator on V .

If λ is an eigenvalue of T , then set

$$E_{\lambda} = \{v \in V \mid T(v) = \lambda v\}, \text{ which}$$

is called the eigenspace of T corresponding to λ . Note E_λ is a subspace of V and consists of all eigenvectors of T with this eigenvalue together with 0 . ⑥

Exercise: $E_\lambda(A) = E_\lambda(L_A)$.

Application: For A as above

and $n \geq 1$ we have $A^n = \begin{pmatrix} -2^{n+2} & 0 & 2^n - 1 \\ 0 & 2^n & 0 \\ -2^{n+1} + 2 & 0 & 2^{n+1} - 1 \end{pmatrix}$

so for example $A^8 = \begin{pmatrix} -254 & 0 & 255 \\ 0 & 256 & 0 \\ -510 & 0 & 511 \end{pmatrix}$

Reason: Set $D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$ so that $Q^{-1}AQ = D$

and so $A = QDQ^{-1}$. Then

$$A^n = A A A \cdots A = \underbrace{(QDQ^{-1})(QDQ^{-1}) \cdots (QDQ^{-1})}_{= I_n}$$

$$= Q D D \cdots D Q^{-1} = Q D^n Q^{-1}$$

$$= \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 2 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2^n & 0 \\ 0 & 0 & 2^n \end{pmatrix} \begin{pmatrix} 2 & 0 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix} \text{ which expands}$$

to the claimed expression for A^n .