

# Lecture 25: Diagonalization and eigenstuff. ①

[§ 5.1 of FIS]

Diagonal matrix:  $A \in M_{n \times n}$  with  $A_{ij} = 0$   
when  $i \neq j$

Ex:  $\begin{pmatrix} 8 & 0 \\ 0 & 2 \end{pmatrix}$

Non Ex:  $\begin{pmatrix} 5 & 3 \\ 3 & 5 \end{pmatrix}$

Linear operator: A linear transformation

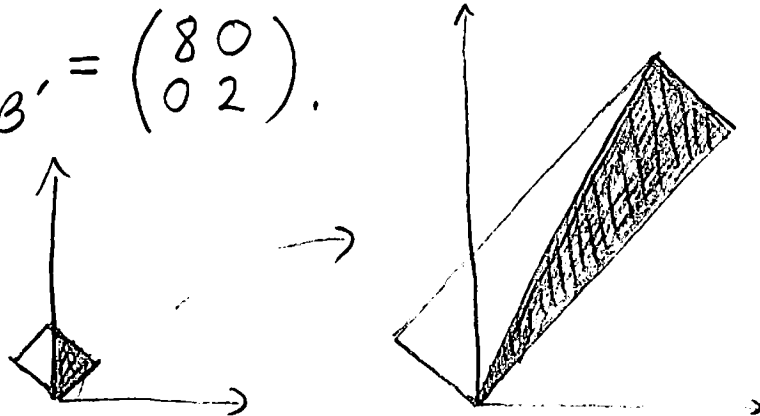
$$T: V \longrightarrow V.$$

Key Def: A linear operator  $T$  of a vector space  $V$  is diagonalizable if there is basis  $\beta$  for  $V$  with  $[T]_{\beta}$  diagonal.

Ex: Set  $A = \begin{pmatrix} 5 & 3 \\ 3 & 5 \end{pmatrix}$  and  $T = L_A$ . In

Lecture 18 saw that if  $\beta' = \{v_1 = (1, 1), v_2 = (-1, 1)\}$

then  $[T]_{\beta'} = \begin{pmatrix} 8 & 0 \\ 0 & 2 \end{pmatrix}$ .



Non Ex:  $B = \begin{pmatrix} 1 & 1 \\ -1 & 3 \end{pmatrix}$  and  $S = L_B$ . ②

Turns out, best you can do is

$$[S]_{\gamma} = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix} \text{ where } \gamma = \{(1, 1), (1, 2)\}$$

[How do we tell the difference? How do we find the right basis?]

If  $\beta = \{v_1, \dots, v_n\}$  is a basis for  $V$  where

$$[T]_{\beta} = \begin{pmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & & \lambda_n \end{pmatrix} \text{ then } T(v_i) = \lambda_i v_i$$

for each  $i$ .

Def: An eigenvector for a linear op  $T$  is a

$v \neq 0$  in  $V$  where there is a scalar  $\lambda$  with

$T(v) = \lambda v$ . The scalar  $\lambda$  is called

the eigenvalue associated to  $v$ .

[Requirement that  $v \neq 0$  is so that  $0$  is not an eigenvector for all  $\lambda$ .]

Ex:  $T = L_A$  with  $A = \begin{pmatrix} 5 & 3 \\ 3 & 5 \end{pmatrix}$ , Then ③

a)  $v = (1, -1)$  is a eigenvector with eigenvalue 2  
as  $T(1, -1) = \begin{pmatrix} 5 & 3 \\ 3 & 5 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \end{pmatrix} = 2 \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ .

b)  $w = (1, 2)$  is not an eigenvector as

$$T(1, 2) = \begin{pmatrix} 5 & 3 \\ 3 & 5 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 11 \\ 13 \end{pmatrix}$$

Thm: A linear operator  $T$  of  $V$  is diagonalizable if and only if there is a basis of  $V$  consisting of eigenvectors of  $T$ .

Pf: ( $\Rightarrow$ ) Clear from above.

( $\Leftarrow$ ) Suppose  $\beta = \{v_1, \dots, v_n\}$  is a basis of  $V$  where  $T(v_i) = \lambda_i v_i$  for some scalar  $\lambda_i$ .

Then

$$[T]_{\beta} = \begin{pmatrix} | & & \\ [T(v_1)]_{\beta} & \cdots & \\ | & & \end{pmatrix} = \begin{pmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & \cdots & \lambda_n \end{pmatrix}$$

Def: A matrix  $A$  is diagonalizable when  $L_A$  is diagonalizable. An eigenvector for  $A$  is one for  $L_A$ .

Alternatively, an eigenvector  $v \in \mathbb{R}^n$  for  $A \in M_{n \times n}$  is one where  $Av = \lambda v$  for some  $\lambda \in \mathbb{R}$ . Also  $A$  is diagonalizable  $\iff$  there exists an invertible  $Q \in M_{n \times n}$  with  $Q^{-1}AQ = (\text{diagonal})$ .

Thm: Suppose  $A \in M_{n \times n}(\mathbb{R})$ . Then  $\lambda \in \mathbb{R}$  is an eigenvalue for  $A$  if and only if  $\det(A - \lambda I_n) = 0$ .

Ex:  $A = \begin{pmatrix} 5 & 3 \\ 3 & 5 \end{pmatrix}$       $A - 2 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 3 & 3 \\ 3 & 3 \end{pmatrix} \xrightarrow{\det} 0$   
 $A - 3 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 3 \\ 3 & 2 \end{pmatrix} \xrightarrow{\det} -5$

[Before proving, we need a lemma.]

Lemma:  $v \in \mathbb{R}^n$  is an eigenvector of  $A \in M_{n \times n}(\mathbb{R})$  (5)  
with eigenvalue  $\lambda \Leftrightarrow v \in \mathcal{N}(A - \lambda I_n)$  and  $v \neq 0$ .

Proof:  $Av = \lambda v \Leftrightarrow Av - \lambda v = 0$   
 $\Leftrightarrow Av - \lambda(I_n v) = 0$   
 $\Leftrightarrow (A - \lambda I_n)v = 0$  ▣

Lemma:  $B \in M_{n \times n}(\mathbb{R})$ . Then  $\mathcal{N}(B) \neq \{0\}$   
if and only if  $\det(B) = 0$ .

Proof: If  $\det(B) \neq 0$  then  $B$  is invertible,  
and so  $Bv = 0 \Rightarrow v = B^{-1}0 = 0$  and so

$\mathcal{N}(B) = \{0\}$ . If instead  $\det(B) = 0$ ,

then  $B$  is not invertible, and so  $\text{rank}(B) < n$   
and  $\text{nullity}(B) > 0$ . In particular

$\mathcal{N}(B) \neq \{0\}$ . ▣

Proof of theorem: By the 1<sup>st</sup> lemma, ⑥

$\lambda$  is an eigenvalue for  $A \iff \mathcal{N}(A - \lambda I_n) \neq \{0\}$ .

$$\iff \det(A - \lambda I_n) = 0.$$

2<sup>nd</sup> Lemma ▣

Def: The characteristic polynomial of  $A \in M_{n \times n}$

is  $f(t) = \det(A - tI_n)$ .

Ex:  $A = \begin{pmatrix} 5 & 3 \\ 3 & 5 \end{pmatrix} \quad A - \begin{pmatrix} t & 0 \\ 0 & t \end{pmatrix} = \begin{pmatrix} 5-t & 3 \\ 3 & 5-t \end{pmatrix}$

Taking det gives

$$\begin{aligned} f(t) &= (5-t)(5-t) - 9 \\ &= t^2 - 10t + 16 = (t-2)(t-8) \end{aligned}$$

Thm: The eigenvalues of  $A$  are exactly the roots of its characteristic polynomial.