

Lecture 22: Properties of the determinant. [§ 4.2 of FIS] ①

Last time: For $A \in M_{n \times n}(\mathbb{R})$ we defined

$$\det(A) = \sum_{j=1}^n (-1)^{j+1} A_{1j} \det(\tilde{A}_{1j})$$

\nwarrow $(n-1) \times (n-1)$ matrix
gotten by deleting
row 1 and column j
from A .

Thm: \det is a linear function
of the r^{th} row when all the
other rows are fixed.

Thm: For $A \in M_{n \times n}(\mathbb{R})$ and $1 \leq r \leq n$ one has

$$\det(A) = \sum_{j=1}^n (-1)^{r+j} A_{rj} \det(\tilde{A}_{rj})$$

Explained last time how the cofactor expansion
thm follows from

Lemma: Let $B \in M_{n \times n}(\mathbb{R})$ for $n \geq 1$. If row
 r of B is e_j , then $\det(B) = (-1)^{r+j} \det(\tilde{B}_{rj})$.

Convention: $\det(0 \times 0 \text{ matrix}) = 1$.

Proof of Lemma: See page 214 of [FIS]. (2)

Similar to proof of first thm in that

we induct on n . Gets a

little messy...

[Discuss alternate viewpoints.]

$$r \begin{pmatrix} A_{11} & \dots & A_{1j} & \dots & A_{1n} \\ \vdots & & \vdots & & \vdots \\ 0 & \dots & 0 & \dots & 0 \\ \vdots & & \vdots & & \vdots \\ 0 & \dots & 0 & \dots & 0 \end{pmatrix}$$

Cor: If A has a row of all zeros, then $\det(A) = 0$.

Pf: Suppose row r of A is 0. As \det is a linear fn of row r when all other rows are fixed, and any linear fn send 0 to 0, we must have $\det(A) = 0$. \square

Cor: If $A \in M_{n \times n}(\mathbb{R})$ has two identical rows, then $\det(A) = 0$.

Proof: Clear for $n=2$, so let's induct on n .

If $s_1 \neq s_2$ are the indices of two matching rows, expand along a row r where $r \neq s_i$:

$$\det(A) = \sum_{j=1}^n (-1)^{r+j} A_{rj} \det(\tilde{A}_{rj})$$

③

↖ also has two identical rows

$$= 0$$

since all $\det(\tilde{A}_{rj}) = 0$ by induction. ▣

[Key to understanding det: row ops.]

How does det change under row ops?

$$\textcircled{1} A \xrightarrow{R_s \leftrightarrow R_s} B \implies \det(B) = -\det(A)$$

Pf: Can assume $s=1$ as any such row op can be implemented by three type $\textcircled{1}$ row ops involving the first row:

$$A \xrightarrow{R_r \leftrightarrow R_1} A' \xrightarrow{R_s \leftrightarrow R_1} A'' \xrightarrow{R_r \leftrightarrow R_1} B$$

↖ $R_r \leftrightarrow R_s$ ↗

Suppose $A = \begin{pmatrix} - & a_1 & - \\ & \vdots & \\ - & a_r & - \end{pmatrix}$ so $B = \begin{pmatrix} - & a_r & - \\ & \vdots & \\ - & a_1 & - \end{pmatrix}$

By corollary, have

(4)

$$0 = \det \begin{pmatrix} -a_1 + a_r- \\ -a_2- \\ \vdots \\ -a_1 + a_r- \end{pmatrix} = \det \begin{pmatrix} -a_1- \\ -a_2- \\ \vdots \\ -a_1 + a_r- \end{pmatrix} + \det \begin{pmatrix} -a_r- \\ -a_2- \\ \vdots \\ -a_1 + a_r- \end{pmatrix}$$

$$= 0 + \det \underbrace{\begin{pmatrix} -a_1- \\ \vdots \\ -a_r- \end{pmatrix}}_A + \det \underbrace{\begin{pmatrix} -a_r- \\ -a_2- \\ \vdots \\ -a_1- \end{pmatrix}}_B + 0$$



$$(2) A \xrightarrow{cR_r} B \implies \det(B) = c \det(A)$$

Pf: Row-wise linearity.

$$(3) A \xrightarrow{kR_s + R_r} B \implies \det(A) = \det(B)$$

Pf: Suppose C is equal to A except row r is replaced by a copy of row s . Then A, B, C are the same except for row r where

$$(\text{row } r \text{ of } B) = (\text{row } r \text{ of } A) + k(\text{row } r \text{ of } C)$$

As $\det(C) = 0$ as it has a repeated row,

$$\text{we get } \det(B) = \det(A) + k \det(C) = \det(A).$$



Can use to compute the determinant.

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$$\underline{\text{Ex:}} \quad A = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 0 & 2 \\ 2 & 1 & 1 \end{pmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{pmatrix} 1 & 0 & 2 \\ 1 & 2 & 3 \\ 2 & 1 & 1 \end{pmatrix} \xrightarrow{-R_1 + R_2}$$

$$\begin{pmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 1 & 1 \end{pmatrix} \xrightarrow{-2R_1 + R_3} \begin{pmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 0 & 1 & -3 \end{pmatrix} \xrightarrow{\frac{1}{2}R_2}$$

$$\begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 1/2 \\ 0 & 1 & -3 \end{pmatrix} \xrightarrow{-R_2 + R_3} \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 1/2 \\ 0 & 0 & -7/2 \end{pmatrix} = C$$

Thm: If A is upper or lower triangular, then $\det(A) =$ product of the diagonal entries.

$$\text{So } \det(C) = -7/2 \text{ and } \det(C) = -\frac{1}{2} \det(A)$$

$$\Rightarrow \det(A) = 7 \text{ as we got last time.}$$

For larger matrices, this is the only way to go: cofactor expansion takes $> n!$ arithmetical operations whereas row reduction takes $C \cdot n^3$.

Even Blue Waters would take $> 10^{100}$ years to compute a 100×100 det by the first method; my laptop took 10^{-2} seconds!

Proof of Thm: If $A = \begin{pmatrix} A_{11} & 0 & 0 & \dots & 0 & 0 \\ A_{21} & A_{22} & 0 & \dots & & \\ A_{31} & A_{32} & A_{33} & 0 & \dots & 0 \\ \vdots & & & & \ddots & \\ A_{n1} & \dots & & & & A_{nn} \end{pmatrix}$

then

$\det(A) = A_{11} \det(\tilde{A}_{11})$
↖ Also lower triangular
 $= \dots$ induct on $n \dots = A_{11} A_{22} \dots A_{nn}$.

For an upper-triangular matrix expand instead on the bottom row.