Lecture 18: Matrices: Inverses and rank

[§2.4 of FIS] and [§MIMN and CRS of B]

Last time: An $n \times n$ matrix A is invertible when there exists an $n \times n$ matrix B with $AB = BA = I_n$.

The inverse B is unique (when it exists) and denoted A^{-1}.

Thm: Suppose $A \in Mat_{n \times n}(\mathbb{R})$. The following are equivalent: ① A is invertible

② The linear trans $L_A : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is invertible.
$x \longmapsto Ax$

③ The nullspace $\mathcal{N}(A) = \{x \in \mathbb{R}^n \mid Ax = 0\}$ is $\{0\}$

Recall: If β is the standard basis of $\mathbb{R}^n$, then $[L_A]_\beta = A$. If B is another $n \times n$ matrix, then $L_{AB} = L_A \circ L_B$.

[The next statement is Theorem 2.5 that you used on the last HW.]

<u>Thm</u>: Suppose $T: V \to W$ is linear with $\dim V = \dim W = n < \infty$. Then T is onto $\Longleftrightarrow$ T is 1-1 $\Longleftrightarrow$ T is invertible.

<u>Reason:</u> $\dim \mathcal{N}(T) + \dim R(T) = \dim V$.

<u>Pf of Thm:</u> ① $\Rightarrow$ ② Assume A is invertible.

Now $L_A \circ L_{A^{-1}} = L_{AA^{-1}} = L_{I_n} = I_{\mathbb{R}^n}$

and $L_A \circ L_{A^{-1}} = L_{A^{-1}A} = L_{I_n} = I_{\mathbb{R}^n}$

so $L_{A^{-1}}$ is the inverse of L_A.

② $\Rightarrow$ ① Let $T: \mathbb{R}^n \to \mathbb{R}^n$ be the inverse of L_A, and set $B = [T]_\beta$ where β = std basis.

Then $I_n = [I_{\mathbb{R}^n}]_\beta = [L_A \circ T]_\beta =$

$[L_A]_\beta [T]_\beta = AB$. Similarly,

$I_n = [T \circ L_A]_\beta = [T]_\beta [L_A]_\beta = BA$.

So A is invertible.

2 $\Leftrightarrow$ 3 By Thm, L_A is invertible

$\Leftrightarrow L_A$ is 1-1 $\Leftrightarrow \mathcal{N}(L_A) = \{0\}$

As $\mathcal{N}(L_A) = \{x \in \mathbb{R}^n \mid \underbrace{L_A(x)}_{Ax} = 0\} = \mathcal{N}(A)$

we have L_A is invertible $\Leftrightarrow \mathcal{N}(A) = \{0\}$. ▨

<u>Computing A^{-1}</u>: Set $A^{-1} = B = \begin{pmatrix} | & & | \\ b_1 & \cdots & b_n \\ | & & | \end{pmatrix}$.

Since $AB = \begin{pmatrix} | & & | \\ Ab_1 & \cdots & Ab_n \\ | & & | \end{pmatrix} = I_n$, we get

$Ab_i = e_i$ for each i. Thus b_i is the sol'n

to $\mathcal{LS}(A, e_i)$.

<u>Ex</u>: Find the inverse of $A = \begin{pmatrix} 4 & 16 & 5 \\ 6 & 25 & 8 \\ 1 & 3 & 1 \end{pmatrix}$.

Consider the "super augmented" matrix

$$\left(\begin{array}{ccc|ccc} 4 & 16 & 5 & 1 & 0 & 0 \\ 6 & 25 & 8 & 0 & 1 & 0 \\ 1 & 3 & 1 & 0 & 0 & 1 \end{array}\right)$$

and do row ops to put in RREF:

$$\begin{array}{c} \left(\begin{array}{ccc:ccc} 1 & 0 & 0 & 1 & -1 & 3 \\ 0 & 1 & 0 & 2 & -1 & -2 \\ 0 & 0 & 1 & -7 & 4 & 4 \end{array}\right) \\ \begin{array}{cccccc} & & & b_1 & b_2 & b_3 \end{array} \end{array}$$

and so $A^{-1} = \begin{pmatrix} 1 & -1 & 3 \\ 2 & -1 & -2 \\ -7 & 4 & 4 \end{pmatrix}$

Note: If A is not invertible, will end up with left-hand part $\neq I_n$ and one of the systems will be inconsistent.

Application: When know A^{-1} can use it to solve any $\mathcal{LS}(A, b)$ since $Ax = b \Rightarrow x = A^{-1}b$.

Any $A \in \mathrm{Mat}_{m \times n}(\mathbb{R})$ has 3 associated subspaces: [We already seen the first two.]

$$\mathcal{N}(A) = \{x \in \mathbb{R}^n \mid Ax = 0\}$$

$$\mathrm{RowSp}(A) = \mathrm{span}(\text{rows of } A) \subseteq \mathbb{R}^n$$

$$\mathrm{ColSp}(A) = \mathrm{span}(\text{cols of } A) \subseteq \mathbb{R}^m$$

In terms of $L_A : \mathbb{R}^n \to \mathbb{R}^m$, $x \mapsto Ax$ have $\mathcal{N}(A) = \mathcal{N}(L_A)$

and $\mathrm{ColSp}(A) = \mathcal{R}(L_A)$ since $\mathcal{R}(L_A)$

is spanned by the image of any basis, in particular by $\{L_A(e_i) = i^{th} \text{ col of } A\}$.

By the Dim Thm applied to L_A get

$$\dim \mathcal{N}(A) + \dim \mathrm{ColSp}(A) = \dim \mathbb{R}^n = n$$

Now, back in Lecture 12 we saw

$$\dim \mathcal{N}(A) + \dim \mathrm{RowSp}(A) = \# \text{ cols of } A = n$$

by using that for a matrix B in RREF one has

$$\dim \mathcal{N}(B) = \# \text{ of non-pivot cols}$$
$$\dim \mathrm{RowSp}(B) = \# \text{ of non-zero rows} = \# \text{ of pivot cols}$$

[Recall that row equivalent matrices have the same null space and row space.]

Thus we get

<u>Thm</u>: $\dim \mathrm{RowSp}(A) = \dim \mathrm{ColSp}(A)$.

This number is called the <u>rank</u> of A.