Math 416: HW 9 due Friday, April 12, 2024.

Webpage: http://dunfield.info/416

Office hours: Wednesday 2:30–3:30pm and Thursday 2:00–3:00pm; other times possible by appointment. My office is 378 Altgeld.

Problems:

1. Section 5.3 of [FIS], Problem 1 parts (c–j).

2. Prove all of the following results:

   Theorem. Let M be an $n \times n$ matrix having real nonnegative entries, let v be a column vector in $\mathbb{R}^n$ having nonnegative coordinates, and let $u \in \mathbb{R}^n$ be the column vector all of whose entries are 1. Then:

   (a) M is a transition matrix if and only if $u^t M = u^t$.

   (b) v is a probability vector if and only if $u^t v = (1)$.

   Corollary.

   (a) The product of two $n \times n$ transition matrices is also a transition matrix. Consequently, any power of a transition matrix is a transition matrix.

   (b) The product of a transition matrix and a probability vector is again a probability vector.

   Note: This is Theorem 5.14 of the 5th edition of [FIS] and Theorem 5.15 of the 4th edition.

3. Prove that if a 1-dimensional subspace W of $\mathbb{R}^n$ contains a nonzero vector with all nonnegative entries, then W contains a unique probability vector.

4. Section 5.3 of [FIS], Problem 7.

5. Section 6.1 of [FIS], Problem 3.

6. Section 6.1 of [FIS], Problem 8 parts (a) and (b).

7. Section 6.1 of [FIS], Problem 9.

8. Section 6.1 of [FIS], Problem 10 and Problem 15(a).

9. The *conjugate transpose* or *adjoint* of $A \in M_{n\times n}(\mathbb{C})$ is the matrix A^* where $(A^*)_{ij} = \overline{(A_{ji})}$. That is, A^* is the result of taking the complex conjugates of the entries of A^t. Like the transpose, it satisfies $(A+B)^* = A^* + B^*$ and $(AB)^* = B^*A^*$. Define the Frobenius inner product on $M_{n\times n}(\mathbb{C})$ by $\langle A, B\rangle = \operatorname{tr}(B^*A)$, where $\operatorname{tr}(C)$ is the sum of the diagonal entries of C. Prove that this is really an inner product by checking the four axioms. *Hint:* To show $\langle A, A\rangle \neq 0$ when $A \neq 0$ observe that the diagonal entires of A^*A are the standard inner product of the columns of A.