## Math 416: //DRAFT ONLY// HW 7 due Friday, March 29, 2024.

**Webpage:** http://dunfield.info/416

**Office hours:** Wednesday 2:30–3:30pm and Thursday 2:00–3:00pm; other times possible by appointment. My office is 378 Altgeld.

**Problems:**

1. Prove the following result that was used in class. Suppose $E$ is the elementary matrix obtained from $I_n$ by the row operation $R$, that is, $I_n \xrightarrow{R} E$. Prove that for all $A \in M_{n\times n}(\mathbb{R})$ one has $A \xrightarrow{R} EA$. Said another way, left-multiplication by $E$ implements the row operation that built $E$ in the first place.

2. Prove that if $A, B \in M_{n\times n}(\mathbb{R})$ are similar matrices then $\det(A) = \det(B)$.

3. A matrix $Q \in M_{n\times n}(\mathbb{R})$ is called orthogonal if $QQ^t = I_n$.

   (a) Prove that if $Q$ is orthogonal then $\det(Q) = \pm 1$.

   (b) Give examples of orthogonal matrices for $n = 2$ with both possible values of the determinant.

4. Suppose $A, B \in M_{n\times n}(\mathbb{R})$ satisfy $AB = I_n$.

   (a) Use the determinant to prove that $A$ is invertible.

   (b) Prove or disprove: $B = A^{-1}$.

5. Section 5.1 of [FIS], Problem 3 parts (a) and (c) if using the 5th edition, or Problem 2 parts (a) and (c) if using the 4th. (The problem is about computing $[T]_\beta$ and determining whether $\beta$ consists of eigenvectors.)

6. Let $T$ be a linear operator on a finite-dimensional vector space $V$.

   (a) Show that $T$ is invertible if and only if 0 is not an eigenvalue of $T$.

   (b) If $T$ is invertible, show that $\lambda^{-1}$ is an eigenvalue of $T^{-1}$ if and only if $\lambda$ is an eigenvalue of $T$.

7. Suppose $T\colon V \to V$ is a linear operator with $V$ finite-dimensional. Suppose $v \in V$ is an eigenvector of $T$ with eigenvalue $\lambda$. As usual, $T^m\colon V \to V$ denotes composition of $T$ with itself $m$ times. Prove that $v$ is also an eigenvector for $T^m$ and give a formula for the corresponding eigenvalue.

8. Section 5.1 of [FIS]. Problem 4(a) if using the 5th edition, but Problem 3(a) if using the 4th. (This problem is about $A = \left(\begin{smallmatrix} 1 & 2 \\ 3 & 2 \end{smallmatrix}\right)$.)

9. Section 5.1 of [FIS], Problem 5 parts (b) and (h) if using the 5th edition, but Problem 4 parts (b) and (h) if using the 4th. (The problem is about finding the eigenvalues of $T$ and a basis $\beta$ for $V$ such that $[T]_\beta$ is diagonal. Part (b) is about $\mathbb{R}^3$ and $T(a, b, c) = (7a - 4b + 10c, \ldots)$ and (h) is about $M_{2\times 2}(\mathbb{R})$).