Math 416: HW 3 due Friday, February 9, 2024.

Webpage: http://dunfield.info/416

Office hours: Wednesday 2:30–3:30pm and Thursday 2:00–3:00pm; other times possible by appointment. My office is 378 Altgeld.

Textbooks: In the assignment, the two texts are abbreviated as follows:

[FIS] Freidberg, Insel, Spence, *Linear Algebra*, 4th or 5th edition, 2002 or 2019.

[B] Breezer, *A First Course in Linear Algebra*, Version 3.5, 2015.

Problems:

1. (a) Suppose A is an $m \times n$ matrix with $m < n$. Show that the null space $\mathcal{N}(A)$ contains a nonzero vector by an argument involving the reduced row echelon form of A.

   (b) Use part (a) to prove that any j vectors in $\mathbb{R}^k$ are linearly dependent if $j > k$.

2. (a) Suppose S is a subset of a vector space V. Show that if $v \in V$ is contained in span(S), then span$(S) =$ span$(S \cup \{v\})$.

   (b) From problem 2(c) on the last HW, consider $V = \mathbb{R}^2$ and $S = \{(x, y) \mid x \geq 0 \text{ and } y \geq x\}$. Use part (a) to give a short proof that span$(S) = \mathbb{R}^2$ by showing that span(S) contains the vectors $(1, 0)$ and $(0, 1)$.

3. Let u and v be distinct vectors in a vector space V. Show that $\{u, v\}$ is linearly dependent if and only if one of u or v is a scalar multiple of the other.

4. Either prove or give a counterexample to the following statement: If u_1, u_2, u_3 are three vectors in $\mathbb{R}^3$ none of which is a scalar multiple of another, then they are linearly independent.

5. In the vector space $\mathcal{F}(\mathbb{R}, \mathbb{R})$ consider the elements $f(t) = \sin(t)$ and $g(t) = \cos(t)$. Is the subset $\{f, g\}$ linearly dependent or linearly independent? Prove your answer.

6. Section 1.6 of [FIS], Problem 1.

7. Section 1.6 of [FIS], Problem 2, parts (a) and (b).

8. Section 1.6 of [FIS], Problem 8.

9. Recall from HW 1 that the subset U of all upper triangular matrices in $M_{n\times n}(\mathbb{R})$ forms a subspace. Find a basis for U and use it to compute the dimension of U.

10. Suppose W is a subspace of a finite-dimensional vector space V. For some $v \in V$ not in W, set $X = \text{span}\,(W \cup \{v\})$. Prove that $\dim(X) = \dim(W) + 1$.