Math 416: HW 2 due Friday, February 2, 2024.

Webpage: http://dunfield.info/416

Office hours: Wednesday 2:30–3:30pm and Thursday 2:00–3:00pm; other times possible by appointment. My office is 378 Altgeld.

Textbooks: In the assignment, the two texts are abbreviated as follows:

[FIS] Freidberg, Insel, Spence, *Linear Algebra*, 4th or 5th edition, 2002 or 2019.

[B] Breezer, *A First Course in Linear Algebra*, Version 3.5, 2015.

Problems:

1. In these questions, you will determine whether one vector is a linear combination of two others.

   (a) Section 1.4 of [FIS], Problem 3: parts (a) and (c).

   (b) Section 1.4 of [FIS], Problem 4: parts (a) and (e).

2. In class, I defined the span of a finite list of vectors $u_1, u_2, \ldots, u_n$. More generally, given a nonempty subset S of a vector space V, one defines $\operatorname{span}(S)$ to be the set of all linear combinations of vectors in S. Here are some problems about the span.

   (a) Section 1.4 of [FIS], Problem 5: parts (g) and (h).

   (b) Suppose S_1 and S_2 are subsets of a vector space V. Show that if S_1 is contained in S_2, then $\operatorname{span}(S_1)$ is contained in $\operatorname{span}(S_2)$.

   (c) Let $V = \mathbb{R}^2$ and $S = \{(x, y) \text{ where } x \geq 0 \text{ and } y \geq x\}$. Find $\operatorname{span}(S)$.

3. Solve each of the following linear systems by writing down its augmented matrix, doing row operations to get a matrix in reduced row echelon form, and using that to find all of the solutions. You should label your row operations as in §RREF of [B].

   (a)

$$\begin{aligned} 2x_1 + x_2 &= 0 \\ x_1 + x_2 &= 1 \\ 3x_1 + 4x_2 &= 5 \\ 3x_1 + 5x_2 &= 7 \end{aligned}$$

(b)

$$
\begin{aligned}
y_1 + 2y_2 - y_3 &= 1 \\
y_1 + y_2 + 2y_3 &= 0 \\
5y_1 + 8y_2 + y_3 &= 1
\end{aligned}
$$

(c)

$$
\begin{aligned}
2x_1 + 4x_2 + 5x_3 + 7x_4 &= 18 \\
x_1 + 2x_2 + x_3 - x_4 &= 3 \\
4x_1 + 8x_2 + 7x_3 + 5x_4 &= 24
\end{aligned}
$$

4. Suppose that A, B, and C, are $m \times n$ matrices with real coefficients. Prove the following three facts from the definition of row equivalence.

   (a) A is row equivalent to A.

   (b) If A is row equivalent to B, then B is row equivalent to A.

   (c) If A is row equivalent to B, and B is row equivalent to C, then A is row equivalent to C.

   Note: A relationship that satisfies these three properties is known as an *equivalence relation*; this is a formal way of saying that a relationship behaves like equality, without requiring the relationship to be as strict as equality itself.

5. Suppose A is an $m \times n$ matrix with real entries. The *null space* of A, denoted $\mathcal{N}(A)$, is the set of all solutions in $\mathbb{R}^n$ to the linear system $\mathcal{LS}(A, 0)$, where here 0 is the zero vector in $\mathbb{R}^m$. Prove that $\mathcal{N}(A)$ is a subspace of $\mathbb{R}^n$.