Thursday, December 5 ∗∗ *Stokes' Theorem*

1. Let S be the portion of the cylinder of radius 2 about the x-axis where $-1 \leq x \leq 1$.

   (a) Draw a picture of S and compute its area without doing any integrals. Hint: How could you make this cylinder out of paper?

   (b) Find a parameterization $\mathbf{r}(u, v)$ of S.

   (c) Does the normal vector field associated to your parameterization point into or out of S? First, try to determine this without doing any calculations, and then check your answer by evaluating $\mathbf{r}_u \times \mathbf{r}_v$.

   (d) If necessary, change your parameterization so that the normal vector field points *inwards*.

   (e) Now consider the vector field $\mathbf{F} = \langle -z,\ xz,\ -xy \rangle$. Compute $\operatorname{curl}\mathbf{F}$.

   (f) Check that $\operatorname{curl}\mathbf{F}$ is the sum of $\mathbf{G} = \langle -2x,\ -1,\ 0 \rangle$ and $\mathbf{H} = \langle 0,\ y,\ z \rangle$.

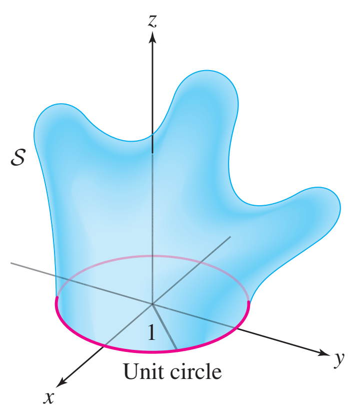
   (g) Use geometric arguments to determine whether the flux of $\mathbf{G}$ is positive, zero, or negative. Remember that we have oriented S so that the normals point inwards. Do the same for $\mathbf{H}$ and $\operatorname{curl}\mathbf{F}$.

   (h) Using your parametrization, directly compute the flux of $\operatorname{curl}\mathbf{F}$.

   (i) Check your answer in (h) using Stokes' Theorem. Note here that ∂S has two boundary components, and make sure that your orient them correctly.

   (j) Check your answer in (h) a second time by using what you learned in (g) to compute the flux of $\mathbf{G}$ and $\mathbf{H}$.

2. Consider the surface S shown below, which is oriented using the outward pointing normal.



   (a) Suppose $\mathbf{F}$ is a vector field on $\mathbb{R}^3$ which is equal to $\operatorname{curl}\mathbf{G}$ for some unknown vector field $\mathbf{G}$. Suppose the line integral of $\mathbf{G}$ around the unit circle (oriented counter-clockwise) in the xy-plane is 25. Determine the flux of $\mathbf{F}$ through S.

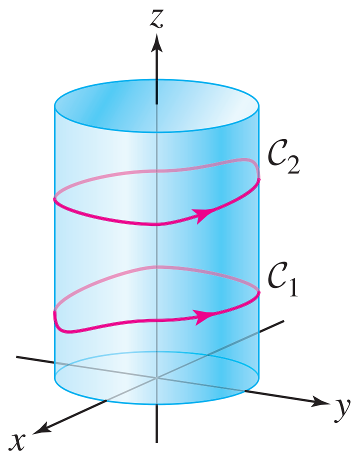
   (b) Suppose $\mathbf{H}$ is a vector field on $\mathbb{R}^3$ which is equal to $\operatorname{curl}\mathbf{B}$ for some unknown vector field $\mathbf{B}$. If $\mathbf{H}(x, y, 0) = \mathbf{k}$, find the flux of $\mathbf{H}$ through the surface S.

Check your answers with the instructor.

3. Let $\mathbf{F} = \langle y^2,\ x^2,\ z^2 \rangle$. Show that

$$\int_{C_1} \mathbf{F} \cdot d\mathbf{r} = \int_{C_2} \mathbf{F} \cdot d\mathbf{r}$$

for any two closed curves as shown lying on a cylinder about the z-axis.



4. Consider the surface T which is the intersection of the plane $x+2y+3z=1$ with the first octant.

   (a) Draw a picture of T.

   (b) Use Stokes' Theorem to evaluate $\int_{\partial T} \mathbf{F} \cdot d\mathbf{r}$ for $\mathbf{F} = \langle y,\ -2z,\ 4x \rangle$. Here, you should orient ∂T counterclockwise when viewed from $(2,2,2)$.

5. Carefully explain how Green's Theorem is actually a special case of Stokes' Theorem.

6. Work the following problem.

20. The magnetic field $\mathbf{B}$ due to a small current loop (which we place at the origin) is called a **magnetic dipole** (Figure 18). Let $\rho = (x^2 + y^2 + z^2)^{1/2}$. For ρ large, $\mathbf{B} = \text{curl}(\mathbf{A})$, where

$$\mathbf{A} = \left\langle -\frac{y}{\rho^3}, \frac{x}{\rho^3}, 0 \right\rangle$$

(a) Let $\mathcal{C}$ be a horizontal circle of radius R with center $(0, 0, c)$, where c is large. Show that $\mathbf{A}$ is tangent to $\mathcal{C}$.
(b) Use Stokes' Theorem to calculate the flux of $\mathbf{B}$ through $\mathcal{C}$.

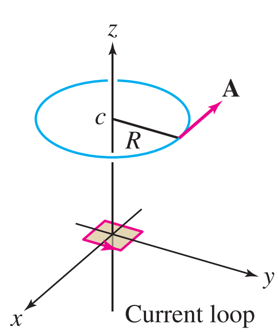


FIGURE 18